Electromagnetic radiation.

Homework

1. It can be shown that the following electric and magnetic fields,

$$E_r = ke^{i(\omega t - kr)}\left[-\frac{1}{(kr)^2} + \frac{i}{(kr)^3}\right]\cos\theta,$$
$$E_\theta = \frac{1}{2}ke^{i(\omega t - kr)}\left[-\frac{i}{kr} - \frac{1}{(kr)^2} + \frac{i}{(kr)^3}\right]\sin\theta,$$
$$H_\psi = \frac{1}{2}k\sqrt{\frac{\varepsilon_0}{\mu_0}}e^{i(\omega t - kr)}\left[-\frac{i}{kr} - \frac{1}{(kr)^2}\right]\sin\theta,$$
$$E_\psi = H_r = H_\theta = 0,$$

written in spherical polar coordinates (r, θ, ψ), with $\omega/k = 1/\sqrt{\varepsilon_0\mu_0}$, satisfy Maxwell's equations in free space in the absence of charges and currents.

(a) Consider an oscillating electric dipole of moment $\boldsymbol{P}e^{i\omega t}$, taking $\boldsymbol{P}$ along the z-axis. Show by considering the near-field ($kr \ll 1$) limit that the fields due to the oscillating electric dipole can be obtained by multiplying the above solution of Maxwell's equations by

$$-\frac{ik^2 P}{2\pi\varepsilon_0}.$$

(b) Obtain an expression valid in the far-field ($kr \gg 1$) limit for the time-averaged Poynting vector due to such an oscillating electric dipole at the origin and show that the total flux of radiation outward through a sphere is given by

$$\frac{\mu_0\sqrt{\varepsilon_0\mu_0}\omega^4 P^2}{12\pi}.$$

[Exam 2003 Question 7]

2. (a) By making a suitable expansion of the differentials $d[(\boldsymbol{b}\cdot\boldsymbol{r})\boldsymbol{r}]$ and $\boldsymbol{b}\times(\boldsymbol{r}\times d\boldsymbol{r})$, or otherwise, prove that

$$\oint_C (\boldsymbol{b}\cdot\boldsymbol{r})d\boldsymbol{r} = \boldsymbol{M}\times\boldsymbol{b},$$

where

$$\boldsymbol{M} \equiv \frac{1}{2}\oint_C \boldsymbol{r}\times d\boldsymbol{r},$$

C is a closed curve, and $\boldsymbol{b}$ is a constant vector. If C is a plane curve, show that the magnitude of $\boldsymbol{M}$ is equal to the area enclosed by C, and that the direction of $\boldsymbol{M}$ is perpendicular to the plane of C, and related to the right-hand screw to the direction in which the curve is traversed.

(b) A circular loop of wire of radius a carries an alternating current $I = I_0\cos\omega t$, where I_0 and ω are constants and $\omega a \ll c$, c being the velocity of light. The loop lies in the x-y plane and is centred upon the origin. Show that, correct to first-order in $1/r$, where r is the distance from the origin, the (complex) vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{i\mu_0 I_0 a^2 k}{4}\frac{e^{i(\omega t - kr)}}{r}\sin\theta\,\hat{\boldsymbol{\psi}},$$

where $k \equiv \omega/c$ and (r, θ, ψ) are the polar coordinates of $\boldsymbol{r}$. Hence, determine the magnetic field $\boldsymbol{H}$ and the electric field $\boldsymbol{E}$, correct to first order in $1/r$, and the corresponding Poynting vector $\boldsymbol{S}$.

3. (a) Show, from Maxwell's equations, how the $\boldsymbol{B}$ and $\boldsymbol{E}$ fields may be parameterised in terms of a vector potential $\boldsymbol{A}$ and a scalar potential ϕ.

   (b) In Lorenz gauge (i.e., $\nabla \cdot \boldsymbol{A} + \varepsilon_0\mu_0 \partial\phi/\partial t = 0$), the vector potential $\boldsymbol{A}$ at a large distance r from a finite system of charges is approximately

$$\boldsymbol{A} = \frac{\mu_0}{4\pi r}\dot{\boldsymbol{P}}(t - r/c),$$

   where $\boldsymbol{P}(t)$ is the dipole moment of the system, $c = 1/\sqrt{\varepsilon_0\mu_0}$ and $\dot{\boldsymbol{P}}(t) \equiv d\boldsymbol{P}/dt$.
   For

$$\boldsymbol{P}(t) = \boldsymbol{P}_0 e^{i\omega t},$$

   where $\boldsymbol{P}_0$ and ω are constants, show that, correct to first order in $1/r$, the corresponding magnetic and electric fields are

$$\boldsymbol{B} = \frac{\omega\mu_0 k}{4\pi}\frac{e^{i(\omega t - kr)}}{r}\hat{\boldsymbol{r}} \times \boldsymbol{P}_0,$$
$$\mathbf{E} = \frac{\omega^2\mu_0}{4\pi}\frac{e^{i(\omega t - kr)}}{r}\hat{\boldsymbol{r}} \times (\boldsymbol{P}_0 \times \hat{\boldsymbol{r}}),$$

   where $k = \omega/c$ and $\hat{\boldsymbol{r}}$ is a unit vector in the direction of $\boldsymbol{r}$.
   Hint: in deriving the result, the following vector idenities may be of use:

$$\nabla \times (f\boldsymbol{a}) = f(\nabla \times \boldsymbol{a}) + \nabla f \times \boldsymbol{a},$$
$$\nabla \cdot (f\boldsymbol{a}) = f(\nabla \cdot \boldsymbol{a}) + \nabla f \cdot \boldsymbol{a},$$
$$\nabla(fg) = f\nabla g + g\nabla f.$$

   (c) A rigid electric dipole p is centred on the origin and lies in the x-y plane. The dipole rotates anticlockwise with constant angular velocity ω, and is along the x axis at $t = 0$. Show that, correct to first order in $1/r$, the $\boldsymbol{B}$ and $\boldsymbol{E}$ fields created by the dipole are:

$$\boldsymbol{B} = \frac{p\omega\mu_0 k}{4\pi r}[\cos(\omega t - kr - \psi)\cos\theta\,\hat{\boldsymbol{\psi}} - \sin(\omega t - kr - \psi)\,\hat{\boldsymbol{\theta}}],$$
$$\boldsymbol{E} = \frac{pk^2}{4\pi\varepsilon_0 r}[\sin(\omega t - kr - \psi)\,\hat{\boldsymbol{\psi}} + \cos(\omega t - kr - \psi)\cos\theta\,\hat{\boldsymbol{\theta}}],$$

   where (r, θ, ψ) are spherical polar coordinates.
   Find the corresponding Poynting vector.

Further Examples

1. (a) A set of point charges is in motion in a finite region surrounding the origin. If the speeds of the charges are much smaller than the velocity of light c, show that at large distance r from the origin the vector potential $\boldsymbol{A}$ is given by

$$\boldsymbol{A} \simeq \frac{\mu_0}{4\pi r}\dot{\boldsymbol{P}}(t - r/c),$$

   where $\boldsymbol{P}(t)$ is the dipole moment of the system at time t.

   (b) For $\boldsymbol{P}(t) = \boldsymbol{P}_0 e^{i\omega t}$, where $\boldsymbol{P}_0$ and ω are constants, find, correct to first order in $1/r$, the corresponding electric and magnetic fields at large r.

Answers:

2(b). $\boldsymbol{H} = -\frac{I_0 a^2 k^2}{4r}e^{i(\omega t - kr)}\sin\theta\,\hat{\boldsymbol{\theta}}$, $\boldsymbol{E} = -\frac{\mu_0\omega I_0 a^2 k}{4r}e^{i(\omega t - kr)}\sin\theta\,\hat{\boldsymbol{\phi}}$.