AMA3001 Electromagnetic Theory

Problem sheet 10

SOLUTIONS

(1)
$$E_r = k e^{i(\omega t - kr)} \left[-\frac{1}{(kr)^2} + \frac{i}{(kr)^3} \right] \cos\theta \quad , \qquad (1)$$

$$E_\theta = \frac{1}{2} k e^{i(\omega t - kr)} \left[-\frac{i}{kr} - \frac{1}{(kr)^2} + \frac{i}{(kr)^3} \right] \sin\theta \quad , \qquad (2)$$

$$H_\varphi = \frac{1}{2} k \sqrt{\frac{\varepsilon_0}{\mu_0}} e^{i(\omega t - kr)} \left[-\frac{i}{kr} - \frac{1}{(kr)^2} \right] \sin\theta \quad , \qquad (3)$$

$$E_\varphi = H_r = H_\theta = 0 \ .$$

(a) The electrostatic potential of the dipole $\vec{p}$ is

$$\varphi = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2} \ .$$

The corresponding electric field is

$$\vec{E} = -\vec{\nabla}\varphi = -\frac{1}{4\pi\varepsilon_0} \left(\hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\hat{\varphi}\frac{\partial}{\partial\varphi} \right) \frac{p\cos\theta}{r^2}$$

$$= -\frac{p}{4\pi\varepsilon_0} \left(\hat{r}\,\frac{-2\cos\theta}{r^3} + \hat{\theta}\,\frac{-\sin\theta}{r^3} \right), \qquad (4)$$

and it behaves as $\frac{1}{r^3}$. Looking at the $\sim\frac{1}{r^3}$ terms in (1) and (2) we have:

$$E_r \simeq \frac{ik}{(kr)^3} \cos\theta \, e^{i(\omega t - kr)} \qquad (5)$$

$$E_\theta \simeq \frac{ik}{2(kr)^3} \sin\theta \, e^{i(\omega t - kr)} \qquad (6)$$

Setting $p = P e^{i\omega t}$ in (4) we have the components:

$$E_r = \frac{P\cos\theta}{2\pi\varepsilon_0 r^3} e^{i\omega t} \quad \text{and} \quad E_\theta = \frac{P\sin\theta}{4\pi\varepsilon_0 r^3} e^{i\omega t}$$

Comparing these with (5) and (6) (and assuming that $kr \ll 1$, so that $e^{i(\omega t - kr)} \simeq e^{i\omega t}$, we see that (5) and (6) would behave as the field of the dipole if they are multiplied by the factor

$$-\frac{i k^2 p}{2\pi \varepsilon_0}. \tag{7}$$

This means that the <u>field of the oscillating dipole</u> can be obtained from (1), (2) and (3) by multiplying them by (7):

$$E_r = -\frac{i k^3 p}{2\pi \varepsilon_0} e^{i(\omega t - kr)} \left[-\frac{1}{(kr)^2} + \frac{i}{(kr)^3} \right] \cos\theta, \tag{8}$$

$$E_\theta = -\frac{i k^3 p}{4\pi \varepsilon_0} e^{i(\omega t - kr)} \left[-\frac{i}{kr} - \frac{1}{(kr)^2} + \frac{i}{(kr)^3} \right] \sin\theta, \tag{9}$$

$$H_\varphi = -\frac{i k^3 p}{4\pi \sqrt{\varepsilon_0 \mu_0}} e^{i(\omega t - kr)} \left[-\frac{i}{kr} - \frac{1}{(kr)^2} \right] \sin\theta. \tag{10}$$

(b) In the far-field limit ($kr \gg 1$) the most slowly decreasing terms that determine the electromagnetic field there are $\sim \frac{1}{kr}$ in square brackets in (9) and (10). Hence, in the leading order:

$$E_\theta \simeq -\frac{k^2 p}{4\pi \varepsilon_0 r} \sin\theta\, e^{i(\omega t - kr)} \tag{11}$$

$$H_\varphi \simeq -\frac{k^2 p}{4\pi \sqrt{\varepsilon_0 \mu_0}\, r} \sin\theta\, e^{i(\omega t - kr)}, \tag{12}$$

with the other components decreasing faster, or equal to zero.

Note that according to (11) and (12), the electric and magnetic fields are perpendicular to the direction of propagation of the wave ($\hat{r}$) and to each other. Their magnitudes also agree with the relation between the electric and magnetic fields in a plane wave (see lecture notes):

$$B = \frac{1}{c} E \,. \qquad \Big\} \; c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Indeed, $B = \mu_0 H_\varphi = -\dfrac{k^2 P}{4\pi r}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\sin\theta\, e^{i(\omega t - kr)}$

and $\quad \dfrac{1}{c} E_\theta = -\dfrac{k^2 P}{4\pi r}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\sin\theta\, e^{i(\omega t - kr)}$.

The Poynting vector $\vec{S} = \vec{E} \times \vec{H}$, and its magnitude (averaged over time, i.e., one period of oscillation of the field) can be obtained from the complex fields (11) and (12) as

$$\langle S \rangle = \frac{1}{2} E_\theta H_\varphi^*$$

$$= \frac{1}{2} \frac{k^4 P^2}{16\pi^2 \varepsilon_0 \sqrt{\varepsilon_0 \mu_0}} \frac{1}{r^2} \sin^2\theta$$

The direction of $\vec{S}$ is along the radius

Integrating this energy flux over a sphere of radius r, we find:

$$\int\limits_{\substack{\text{over}\\ \text{sphere}}} \langle S \rangle \, dS = \int_0^{2\pi}\!\!\int_0^{\pi} \langle S \rangle \, r^2 \sin\theta \, d\theta \, d\varphi$$

$$= \frac{k^4 P^2}{32\pi^2 \varepsilon_0 \sqrt{\varepsilon_0 \mu_0}} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\varphi \,.$$

The integrals involved are $\displaystyle\int_0^{2\pi} d\varphi = 2\pi$

and $\int_0^\pi \sin^2\theta \sin\theta \, d\theta = -\int_0^\pi (1-\cos^2\theta)\, d(\cos\theta)$

$$= \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^\pi = -(-1) + \frac{(-1)^3}{3} - \left(-1+\frac{1}{3}\right)$$

$$= 2 - \frac{2}{3} = \frac{4}{3}.$$

Therefore, the energy flux is

$$\int \langle S \rangle \, dS = \frac{k^4 P^2}{32\pi^2 \varepsilon_0 \sqrt{\varepsilon_0 \mu_0}} \, 2\pi \cdot \frac{4}{3}$$

$$= \frac{k^4 p^2}{12\pi \, \varepsilon_0 \sqrt{\varepsilon_0 \mu_0}}.$$

Finally, using $k = \frac{\omega}{c} = \sqrt{\varepsilon_0 \mu_0}\, \omega$,

$$k^4 = \varepsilon_0^2 \mu_0^2 \omega^4,$$

we find the flux as

$$\int \langle S \rangle \, dS = \frac{\mu_0 \sqrt{\varepsilon_0 \mu_0}\, \omega^4 P^2}{12\pi}.$$

(2) (a) For a constant vector $\vec{b}$,

$$d[(\vec{b}\cdot\vec{r})\vec{r}] = (\vec{b}\cdot d\vec{r})\cdot\vec{r} + (\vec{b}\cdot\vec{r})\, d\vec{r} \qquad (1)$$

and
$$\vec{b}\times(\vec{r}\times d\vec{r}) = \vec{r}\,(\vec{b}\cdot d\vec{r}) - d\vec{r}\,(\vec{b}\cdot\vec{r}) \qquad (2)$$

Subtracting (2) from (1),

$$d[(\vec{b}\cdot\vec{r})\vec{r}] - \vec{b}\times(\vec{r}\times d\vec{r}) = 2(\vec{b}\cdot\vec{r})\, d\vec{r},$$

or
$$(\vec{b}\cdot\vec{r})\, d\vec{r} = \frac{1}{2}\left\{ d[(\vec{b}\cdot\vec{r})\vec{r}] - \vec{b}\times(\vec{r}\times d\vec{r})\right\}.$$

Substituting this into the integral over a closed curve C, we have:

$$\oint_C (\vec{b}\cdot\vec{r})\,d\vec{r} = \frac{1}{2}\oint_C d[(\vec{b}\cdot\vec{r})\vec{r}] - \frac{1}{2}\oint_C \vec{b}\times(\vec{r}\times d\vec{r})$$

The first term on the right-hand side is the integral of the total differential, and over a closed curve, it gives zero (since $(\vec{b}\cdot\vec{r})\vec{r}$ at the "start" and "end" of the integration path are the same).

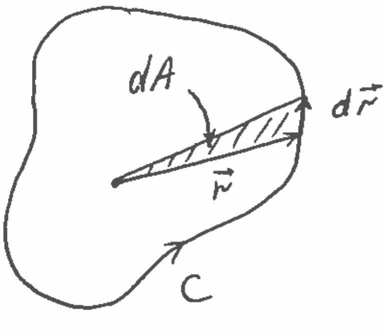
Interchanging the order of vectors in the 2nd integral (and changing the sign), we have:

$$\oint_C (\vec{b}\cdot\vec{r})\,d\vec{r} = \left(\frac{1}{2}\oint_C \vec{r}\times d\vec{r}\right)\times\vec{b}$$

$$= \vec{M}\times\vec{b},$$

where $\vec{M} = \frac{1}{2}\oint_C \vec{r}\times d\vec{r}$, as required.

If C is a planar curve than both $\vec{r}$ and $d\vec{r}$ lie in this plane (assuming that the origin is also in this plane), and $\vec{r}\times d\vec{r}$ is perpendicular to the plane. In the diagram, $\vec{r}\times d\vec{r}$ is towards us if the curve is traversed in the positive (anti-clockwise) direction.



Hence,

$$\frac{1}{2}\vec{r}\times d\vec{r} = \vec{n}\,\frac{1}{2}|\vec{r}\times d\vec{r}|,$$

where $\vec{n}$ is a unit vector perpendicular to the plane (towards us).

$\frac{1}{2}|\vec{r}\times d\vec{r}| = dA$ - area of the elementary triangle with sides $\vec{r}$ and $d\vec{r}$

Hence,

$$\vec{M} = \frac{1}{2} \oint \vec{r} \times d\vec{r} = \vec{n} \oint_c dA = \underline{\underline{\vec{n} A}},$$

where A is the area inside the curve C.

Note that the direction of $\vec{n}$ is that of a right-hand screw rotated in the direction in which C is traversed. (The screw is moving out of the page.)

(b) Consider the retarded vector potential:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{j}(\vec{r}',t')}{|\vec{r}-\vec{r}'|} dV',$$

$$\text{where} \quad t' = t - \frac{|\vec{r}-\vec{r}'|}{c} . \qquad (1)$$

If the potential is due to a current $I(t)$ that flows in a loop C, we replace

$$\vec{j}\, dV' \longrightarrow I\, d\vec{r}'$$

and have the vector potential in the form

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \oint_C \frac{I(t')\, d\vec{r}'}{|\vec{r}-\vec{r}'|} . \qquad (2)$$

Assuming that the origin is close to the loop and considering the retarded potential at distances r much greater than the size of the loop, we have $r' \ll r$ in (1), and can expand:

$$t' = t - \frac{r}{c} - \frac{1}{c}\vec{\nabla} r \cdot (-\vec{r}') + \ldots \qquad \vec{\nabla} r = \frac{\vec{r}}{r}$$

$$\Rightarrow \quad t' \simeq t - \frac{r}{c} + \frac{\vec{r}\cdot\vec{r}'}{cr} + O\left(\frac{1}{r}\right) .$$

Neglecting $\vec{r}'$ in comparison with $\vec{r}$ in the denominator of the integrand in (2) we have:

$$\vec{A}(\vec{r},t) \simeq \frac{\mu_0}{4\pi r} \oint_c I(t')\, d\vec{r}'$$

$$\simeq \frac{\mu_0}{4\pi r} \oint_c I\left(t - \frac{r}{c} + \frac{\vec{r}\cdot\vec{r}'}{cr}\right) d\vec{r}'$$

$$\simeq \frac{\mu_0}{4\pi r}\left[\oint_c I\left(t-\frac{r}{c}\right) d\vec{r}' + \oint_c \dot{I}\left(t-\frac{r}{c}\right)\frac{\vec{r}\cdot\vec{r}'}{cr}\, d\vec{r}'\right]$$

where we have expanded $I(t')$ in Taylor series to first order in $\frac{\vec{r}\cdot\vec{r}'}{cr}$, and $\dot{I}$ is the time derivative.

The first integral is square brackets is zero (since $I\left(t-\frac{r}{c}\right)$ does not depend on $\vec{r}'$ and $\oint_c d\vec{r}' = 0$).

Using the result from part (a), we transform the second integral in square brackets and obtain:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi r}\,\frac{\dot{I}\left(t-\frac{r}{c}\right)}{cr}\oint_c (\vec{r}\cdot\vec{r}')\, d\vec{r}'$$

$$= \frac{\mu_0}{4\pi c r^2}\left[\dot{I}\left(t-\frac{r}{c}\right)\ \frac{1}{2}\oint_c \vec{r}'\times d\vec{r}'\right]\times\vec{r}$$

Recalling the definition of the magnetic dipole moment of a current loop,

$$\vec{m} = \frac{I}{2}\oint_c \vec{r}'\times d\vec{r}' ,$$

we see that for the time-dependent current the vector potential is

$$\vec{A}(\vec{r},t) = \frac{\mu_0\, \dot{\vec{m}}(t-\frac{r}{c}) \times \vec{r}}{4\pi c\, r^2} .$$

This vector potential describes so-called magnetic dipole radiation

In the problem given, $I(t) = I_0 \cos\omega t$,

or, in the complex form, $I(t) = I_0 e^{i\omega t}$.

The corresponding magnetic moment is

$$\vec{m}(t) = I(t)\, \vec{k} A = \pi a^2 I(t)\, \vec{k} ,$$

$A = \pi a^2$ is the area inside the loop

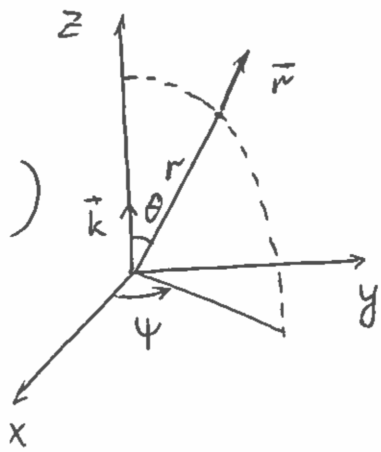
where $\vec{k}$ is along the z axis, so that

$$\vec{m}(t) = \pi a^2 I_0 e^{i\omega t}\, \vec{k}$$

$$\dot{\vec{m}}(t) = i\omega \pi a^2 I_0 e^{i\omega t}\, \vec{k}$$

The corresponding vector potential is

$$\vec{A}(\vec{r},t) = \frac{\mu_0\, i\omega \pi a^2 I_0\, e^{i\omega(t-r/c)}\, \vec{k}\times\vec{r}}{4\pi c\, r^2} \qquad (3)$$



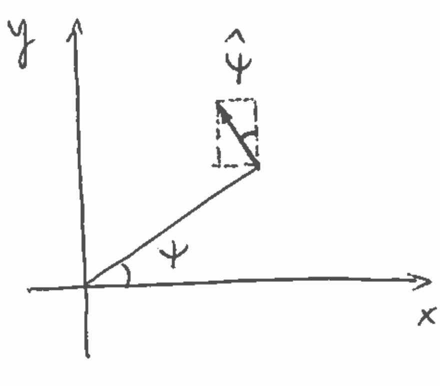
$$\vec{r} = r(\sin\theta\cos\psi\, \vec{i} + \sin\theta\sin\psi\, \vec{j} + \cos\theta\, \vec{k})$$

$$\vec{k}\times\vec{r} = r(\sin\theta\cos\psi\, \underbrace{\vec{k}\times\vec{i}}_{\vec{j}} + \sin\theta\sin\psi\, \underbrace{\vec{k}\times\vec{j}}_{-\vec{i}})$$

$$= r(-\sin\psi\, \vec{i} + \cos\psi\, \vec{j})\sin\theta$$

$$= r\, \hat{\psi}\, \sin\theta ,$$

where $\hat{\psi}$ is the unit vector in the ψ direction in spherical coordinates.



Introducing the wave number

$$k = \frac{\omega}{c}$$

we can write the vector potential (3) as

$$\vec{A}(\vec{r},t) = \frac{i\mu_0 I_0 a^2 k}{4} \frac{e^{i(\omega t - kr)}}{r} \sin\theta \, \hat{\varphi} . \qquad (4)$$

To find the magnetic field, use

$$\vec{H} = \frac{1}{\mu_0}\vec{B} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}$$

$$= \frac{1}{\mu_0}\begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ 0 & 0 & r\sin\theta A_\varphi \end{vmatrix}$$

According to (4), A_φ is the only nonzero component.

$$= \frac{1}{\mu_0}\left\{ \frac{1}{r^2\sin\theta}\hat{r}\frac{\partial}{\partial\theta}(r\sin\theta A_\varphi) - \frac{1}{r\sin\theta}\hat{\theta}\frac{\partial}{\partial r}(r\sin\theta A_\varphi)\right\}$$

$$= \frac{i I_0 a^2 k}{4}\left\{ \frac{1}{r^2\sin\theta}\hat{r}\frac{\partial}{\partial\theta}(\sin^2\theta)\, e^{i(\omega t - kr)} - \frac{1}{r\sin\theta}\hat{\theta}\frac{\partial}{\partial r}\left(e^{i(\omega t-kr)}\right)\sin^2\theta \right\}$$

$$= \frac{i I_0 a^2 k}{4}\left\{ \frac{2\cos\theta}{r^2}\hat{r} + \frac{ik\sin\theta}{r}\hat{\theta}\right\} e^{i(\omega t - kr)}$$

At large distances the leading term is the 2nd term in $\{\ldots\}$, so we have:

$$\vec{H} \simeq -\frac{I_0 a^2 k^2}{4r}\sin\theta\,\hat{\theta}\, e^{i(\omega t - kr)} . \qquad (5)$$

Since the charge density $\rho(\vec{r},t)=0$ in this problem, the corresponding scalar potential is also zero:

$$\varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,dv' = 0,$$

and the electric field is given by

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = +\frac{\mu_0\omega I_0 a^2 k}{4}\,\frac{e^{i(\omega t-kr)}}{r}\sin\theta\,\hat{\varphi}. \quad (6)$$

[Note that we could also find it from the relation between $\vec{E}$ and $\vec{B}$ in a plane wave:

$$\vec{B} = \frac{1}{c}\,\vec{n}\times\vec{E},$$

which gives

$$\begin{aligned}\vec{E} &= -c\,\vec{n}\times\vec{B}\\ &= -\mu_0 c\,\vec{n}\times\vec{H}\\ &= -\mu_0 c\,\hat{r}\times\vec{H},\end{aligned}$$

hence, from (5):

$$\vec{E} = +\frac{\mu_0 c I_0 a^2 k^2}{4r}\sin\theta\,\underbrace{\hat{r}\times\hat{\theta}}_{=\hat{\varphi}}\,e^{i(\omega t-kr)}$$

and since $ck=\omega$, we again obtain (6).]

To find the Poynting vector $\vec{S} = \vec{E}\times\vec{H}$ we need to take real parts of (5) and (6). This gives $\cos(\omega t-kr)$ instead of $e^{i(\omega t-kr)}$ in each of them, so we obtain:

$$\vec{S} = -\frac{\mu_0 \omega I_0^2 a^4 k^3}{16 r^2} \cos^2(\omega t - kr) \sin^2\theta \underbrace{\hat{\varphi} \times \hat{\theta}}_{= -\hat{r}}$$

$$\vec{S} = \frac{\mu_0 \omega k^3 I_0^2 a^4}{16 r^2} \cos^2(\omega t - kr) \sin^2\theta \; \hat{r} \; .$$

[Averaging this over time and integrating over a sphere of radius r allows one to find the total power radiated by the magnetic dipole]

(3) (a) Since $\vec{\nabla} \cdot \vec{B} = 0$, (1)

the magnetic induction $\vec{B}$ can be presented as the curl of a vector field $\vec{A}$ (vector potential)

$$\vec{B} = \vec{\nabla} \times \vec{A} \, ,$$

so that (1) is automatically satisfied.

Then, from Maxwell's 3rd equation,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \, ,$$

we obtain:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} \, ,$$

or

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \, .$$

Hence, $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ can be presented as a gradient of a scalar function, and we introduce the scalar potential φ through

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\varphi ,$$

which gives

$$\underline{\underline{\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}}} .$$

(b) For a system with dipole moment $\vec{P}(t)$ due to charges moving with nonrelativistic speeds, the vector potential at large distance is given by

$$\vec{A} = \frac{\mu_0}{4\pi r} \dot{\vec{P}}(t - \frac{r}{c}) .$$

If $\vec{P}(t) = \vec{P}_0 e^{i\omega t}$, this gives

$$\vec{A} = \frac{i\omega\mu_0}{4\pi r} \vec{P}_0 e^{i\omega(t - r/c)} ,$$

and introducing the wave number $k = \frac{\omega}{c}$,

we have:

$$\vec{A} = \frac{i\omega\mu_0}{4\pi r} \vec{P}_0 e^{i(\omega t - kr)} . \qquad (2)$$

The corresponding magnetic field can be found from $\vec{B} = \vec{\nabla} \times \vec{A}$:

$$\vec{B} = \vec{\nabla} \times \left[\frac{i\omega\mu_0}{4\pi r} \vec{P}_0 e^{i(\omega t - kr)} \right]$$

$$= \frac{i\omega\mu_0}{4\pi} \vec{\nabla} \frac{e^{i(\omega t - kr)}}{r} \times \vec{P}_0$$

$$= \frac{i\omega\mu_0}{4\pi} \left[\vec{\nabla}\left(\frac{1}{r}\right) e^{i(\omega t - kr)} + \frac{1}{r} \vec{\nabla}\left(e^{i(\omega t - kr)}\right) \right] \times \vec{P}_0$$

$$\vec{\nabla}\frac{1}{r} = -\frac{1}{r^2}\vec{\nabla}r = -\frac{1}{r^2}\frac{\vec{r}}{r}.$$

$$\vec{\nabla}e^{i(\omega t-kr)} = e^{i(\omega t-kr)}(-ik)\vec{\nabla}r$$

$$= -ik\, e^{i(\omega t-kr)}\frac{\vec{r}}{r}$$

Thus we see that the leading (more slowly decreasing) term in the expression for $\vec{B}$ is given by the 2nd term in square brackets:

$$\vec{B} \simeq \frac{i\omega\mu_0}{4\pi}\cdot\frac{1}{r}(-ik)e^{i(\omega t-kr)}\,\hat{r}\times\vec{P}_0$$

$$\Rightarrow \quad \vec{B} = \frac{\omega\mu_0 k}{4\pi}\,\frac{e^{i(\omega t-kr)}}{r}\,\hat{r}\times\vec{P}_0 . \qquad (3)$$

At such large distances the electromagnetic field behaves locally as a plane wave, so that $\vec{E}$ and $\vec{B}$ are related by

$$\vec{B} = \frac{1}{c}\,\hat{n}\times\vec{E} = \frac{1}{c}\,\hat{r}\times\vec{E}.$$

Multiplying this on the left with $\hat{r}$:

$$\hat{r}\times\vec{B} = \frac{1}{c}\,\hat{r}\times(\hat{r}\times\vec{E})$$

$$= \frac{1}{c}\Big[\hat{r}\,\underbrace{(\hat{r}\cdot\vec{E})}_{=0} - \vec{E}\,\underbrace{(\hat{r}\cdot\hat{r})}_{=1}\Big]$$

[transversal wave]

$$\Rightarrow \quad \vec{E} = -c\,\hat{r}\times\vec{B}.$$

Using the magnetic field from (3), we obtain:

$$\vec{E} = -\frac{c\omega\mu_0 k}{4\pi}\frac{e^{i(\omega t - kr)}}{r}\,\hat{r}\times(\hat{r}\times\vec{p}_0),$$

and using $ck = \omega$ and swapping the two vectors in the cross product, we have:

$$\vec{E} = \frac{\omega^2\mu_0}{4\pi}\frac{e^{i(\omega t - kr)}}{r}\,\hat{r}\times(\vec{p}_0\times\hat{r}). \qquad (4)$$

[An alternative way of deriving the electric field is to find the scalar potential φ from the Lorenz condition $\vec{\nabla}\cdot\vec{A} + \varepsilon_0\mu_0\frac{\partial\varphi}{\partial t} = 0$, using $\vec{A}$ from equation (2), and then using $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}$. In doing so we can work in leading order, keeping only terms that decrease as $\frac{1}{r}$ at large distances. Ultimately, this leads to (4).]

(c) If the dipole rotates anticlockwise with angular velocity ω, its x and y components can be written as

$$p_x(t) = p\cos\omega t$$
$$p_y(t) = p\sin\omega t.$$

This dipole moment can be written as the real part of $\vec{p}_0 e^{i\omega t}$ if we choose

$$\vec{p}_0 = p\vec{i} - ip\vec{j}. \qquad (5)$$

Indeed: $\mathrm{Re}\left[(p\vec{i} - ip\vec{j})e^{i\omega t}\right]$

$$= p\vec{i}\cos\omega t + p\vec{j}\sin\omega t.$$

To find $\vec{B}$ and $\vec{E}$ we need to substitute (5) into (3) and (4) and take the real parts:

$$\vec{B} = \frac{\omega \mu_0 k}{4\pi} \frac{p}{r} \mathrm{Re}\left[e^{i(\omega t - kr)} \hat{r} \times (\vec{i} - i\vec{j}) \right]$$

$$= \frac{p\omega \mu_0 k}{4\pi r} \left[\cos(\omega t - kr)\, \hat{r} \times \vec{i} + \sin(\omega t - kr)\, \hat{r} \times \vec{j} \right]$$

Unit vectors of the spherical polar coordinates are

$$\hat{r} = \sin\theta\cos\varphi\, \vec{i} + \sin\theta\sin\varphi\, \vec{j} + \cos\theta\, \vec{k} \qquad (A)$$

$$\hat{\theta} = \cos\theta\cos\varphi\, \vec{i} + \cos\theta\sin\varphi\, \vec{j} - \sin\theta\, \vec{k} \qquad (B)$$

$$\hat{\varphi} = -\sin\varphi\, \vec{i} + \cos\varphi\, \vec{j} \qquad (C)$$

Multiplying (A) by $\cos\theta$, (B) by $\sin\theta$ and subtracting them gives

$$\vec{k} = \cos\theta\, \hat{r} - \sin\theta\, \hat{\theta} .$$

Substituting this into (A), rearranging and dividing through by $\sin\theta$, we find

$$\sin\theta\, \hat{r} + \cos\theta\, \hat{\theta} = \cos\varphi\, \vec{i} + \sin\varphi\, \vec{j} \qquad (D)$$

Multiplying this by $\sin\varphi$ and (C) by $\cos\varphi$ and adding them up gives

$$\vec{j} = \sin\theta\sin\varphi\, \hat{r} + \cos\theta\sin\varphi\, \hat{\theta} + \cos\varphi\, \hat{\varphi} .$$

Multiplying (D) by $\cos\varphi$, (C) by $\sin\varphi$ and subtracting the latter from the former gives

$$\vec{i} = \sin\theta\cos\varphi\, \hat{r} + \cos\theta\cos\varphi\, \hat{\theta} - \sin\varphi\, \hat{\varphi} .$$

Therefore,

$$\hat{r}\times\vec{i} = \cos\theta\cos\varphi\,\hat{r}\times\hat{\theta} - \sin\varphi\,\hat{r}\times\hat{\varphi}$$

$$= \cos\theta\cos\varphi\,\hat{\varphi} + \sin\varphi\,\hat{\theta}\,.$$

$$\hat{r}\times\vec{j} = \cos\theta\sin\varphi\,\hat{r}\times\hat{\theta} + \cos\varphi\,\hat{r}\times\hat{\varphi}$$

$$= \cos\theta\sin\varphi\,\hat{\varphi} - \cos\varphi\,\hat{\theta}\,.$$

$\hat{r}\times\hat{\theta} = \hat{\varphi}$
$\hat{\theta}\times\hat{\varphi} = \hat{r}$
$\hat{\varphi}\times\hat{r} = \hat{\theta}$, obtained by circular permutations

Substituting into the last expression for $\vec{B}$ (top of page 15), we find:

$$\vec{B} = \frac{p\omega\mu_0 k}{4\pi r}\Big[\big(\cos(\omega t-kr)\cos\varphi + \sin(\omega t-kr)\sin\varphi\big)\times\cos\theta\,\hat{\varphi} + \big(\cos(\omega t-kr)\sin\varphi - \sin(\omega t-kr)\cos\varphi\big)\,\hat{\theta}\Big]$$

$$= \frac{p\omega\mu_0 k}{4\pi r}\Big[\cos(\omega t-kr-\varphi)\cos\theta\,\hat{\varphi} - \sin(\omega t-kr-\varphi)\,\hat{\theta}\Big]\,.$$

The quickest way of finding $\vec{E}$ now is through

$$\vec{E} = -c\,\vec{n}\times\vec{B} = -c\,\hat{r}\times\vec{B}$$

$$= -\frac{cp\omega\mu_0 k}{4\pi r}\Big[\cos(\omega t-kr-\varphi)\cos\theta\,\hat{r}\times\hat{\varphi} - \sin(\omega t-kr-\varphi)\,\hat{r}\times\hat{\theta}\Big]$$

$$= \frac{cp\omega\mu_0\varepsilon_0 k}{4\pi\varepsilon_0 r}\Big[\sin(\omega t-kr-\varphi)\,\hat{\varphi} + \cos(\omega t-kr-\varphi)\cos\theta\,\hat{\theta}\Big]$$

$$= \frac{pk^2}{4\pi\varepsilon_0 r}\Big[\sin(\omega t-kr-\varphi)\,\hat{\varphi} + \cos(\omega t-kr-\varphi)\cos\theta\,\hat{\theta}\Big].$$

[$c\omega\varepsilon_0\mu_0 = \dfrac{c\omega}{c^2} = \dfrac{\omega}{c} = k$ used here.]

Finally, the Poynting vector is:

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$= \frac{p^2 \omega k^3}{16 \pi^2 \varepsilon_0 r^2} \Big[-\sin^2(\omega t - kr - \psi)\, \hat{\varphi} \times \hat{\theta}$$

$$+ \cos^2(\omega t - kr - \psi) \cos^2\theta \;\hat{\theta} \times \hat{\varphi} \Big]$$

$$= \frac{p^2 \omega k^3}{16 \pi^2 \varepsilon_0 r^2} \left[\sin^2(\omega t - kr - \psi) + \cos^2(\omega t - kr - \psi) \cos^2\theta \right] \hat{r} \,.$$

The pre-factor in this formula can be written in different ways, e.g.

$$\frac{p^2 \omega k^3}{16 \pi^2 \varepsilon_0 r^2} = \frac{p^2 \omega k^3 \mu_0}{16 \pi^2 \varepsilon_0 \mu_0 r^2} = \frac{p^2 \omega k^3 c^2 \mu_0}{16 \pi^2 r^2}$$

$$= \frac{p^2 \omega^3 k \mu_0}{16 \pi^2 r^2} \,,$$

or, substituting $k = \frac{\omega}{c} = \omega \sqrt{\varepsilon_0 \mu_0}$,

$$\frac{p^2 \omega^4 \mu_0 \sqrt{\varepsilon_0 \mu_0}}{16 \pi^2 r^2} \,.$$

As in the case of a linear oscillating dipole, the radiated power is proportional to the 4th power of the frequency ω.