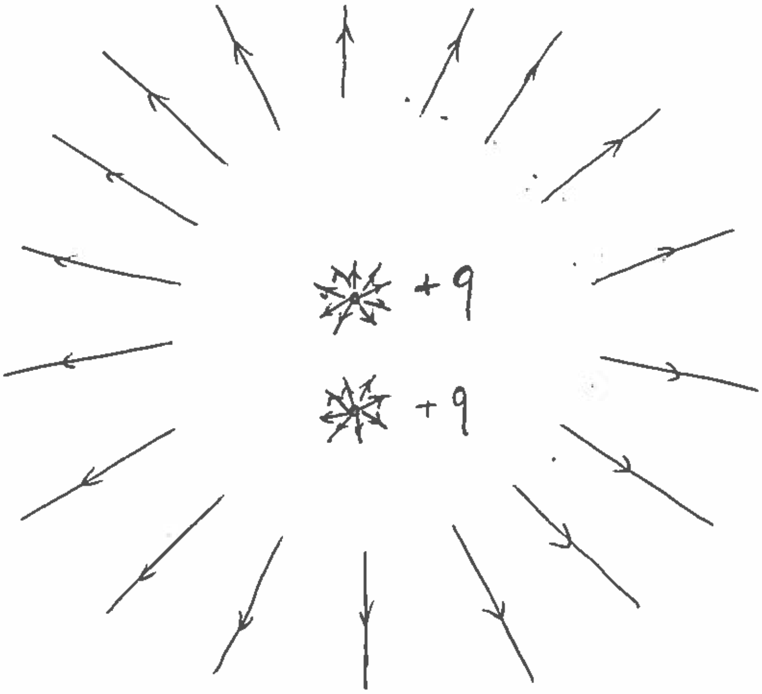
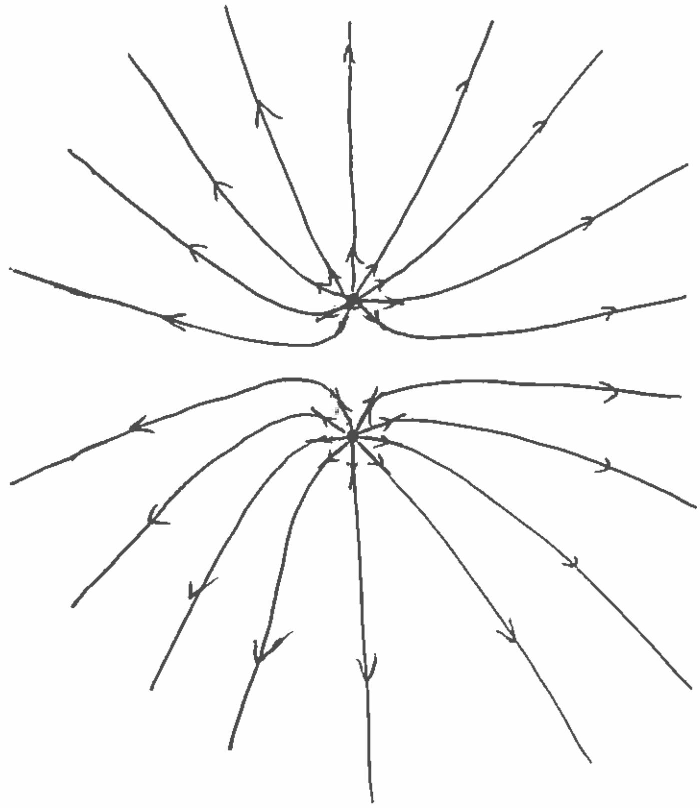
AMA3001 Electromagnetic Theory
Problem Sheet 1
SOLUTIONS

(1) (a) The field near the charges is similar to that of an isolated positive charge, while far away from the charges the field is similar to that of charge $2q$.

Initial sketch:



Final version:

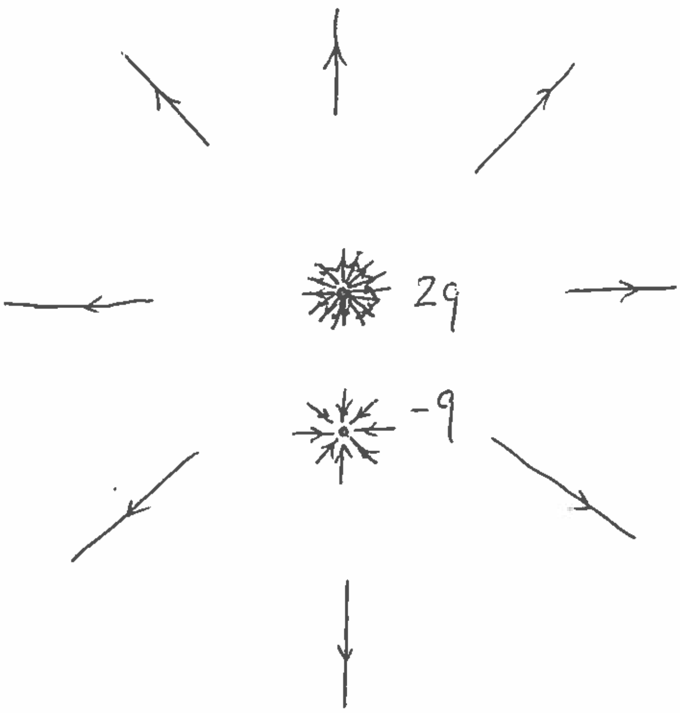


Note: the total number of lines emerging from the charges and that going towards infinity must be the same, since the lines do not end anywhere except at ∞ (or at some negative charges, which we do not have here)

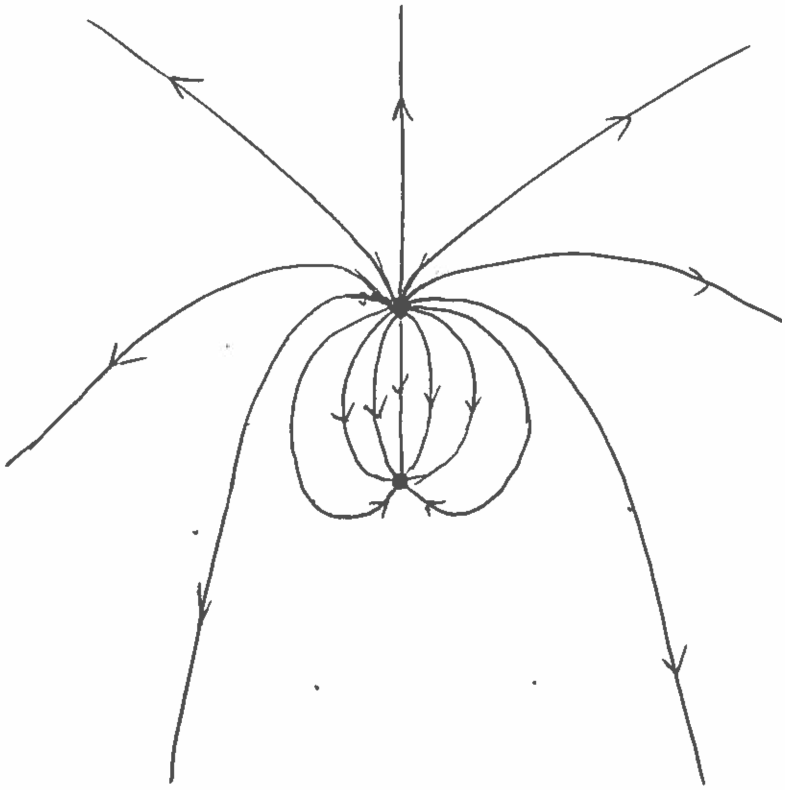
Note: the field at the midpoint between the charges is exactly zero. Hence, there is no line that passes through this point, as otherwise it would have no tangent.

(b) In this case there are twice as many lines that emerge from $2q$ as those that go into $-q$. Hence, the lines emerging from $2q$ and not ending up at $-q$ must radiate out towards infinity. At large distances the field lines are similar to those of charge q ($=2q-q$)

Initial sketch:



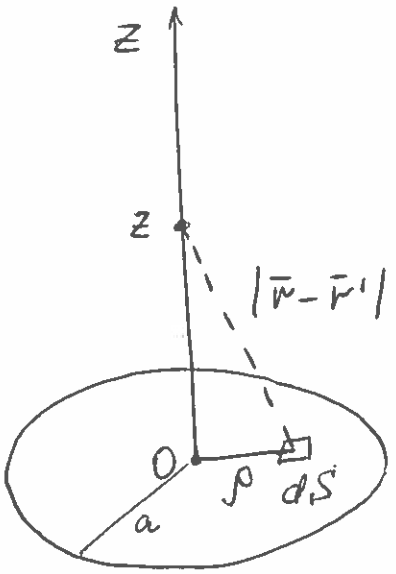
Final sketch:



The lines that emerge from $2q$ into the lower hemisphere (i.e. lower semicircle in the diagram) will end up leading towards $-q$, while the lines that emergy into the top hemisphere (semicircle) will escape towards infinity.

In this case I have shown 14 field lines emerging from $2q$, with 7 of them ending on $-q$, and the other 7 going towards infinity. Note that there is a point below charge $-q$ at which the repulsion from $2q$ and attraction to $-q$ cancel. Hence, there should be no line going through this point, as $E=0$ there.

(2)



The charge on dS is $dq = \sigma dS$. Its contribution to the potential at point z is

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\,\frac{dq}{|\bar{r}-\bar{r}'|},$$

where $|\bar{r}-\bar{r}'| = \sqrt{z^2+\rho^2}$ (see diag.)

$$\Rightarrow \quad \varphi(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dS}{|\bar{r}-\bar{r}'|}$$

{ Here the integral is over the disk }

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{a} \frac{\rho\, d\rho\, d\psi}{\sqrt{\rho^2+z^2}}$$

{ $dS = \rho\, d\rho\, d\psi$ in plane polar coordinates on the disk }

$$= \frac{\sigma}{4\pi\varepsilon_0}\,\underset{\substack{\uparrow\\ \text{integral}\\ \text{over } \psi}}{2\pi}\int_0^{a} \frac{\frac{1}{2}\,d(\rho^2+z^2)}{\sqrt{\rho^2+z^2}}$$

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C$$

$$= \frac{\sigma}{2\varepsilon_0}\left[\frac{1}{2}\, 2\sqrt{\rho^2+z^2}\right]_0^a$$

[$d(\rho^2) = 2\rho\, d\rho$ have been used]

$$= \underline{\frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+z^2} - z\right)},$$ as required.

The electric field on the z axis is along this axis due to symmetry. Hence, $\vec{E} = E_z \vec{k} = -\frac{\partial\varphi}{\partial z}\vec{k}$, and $E_z = -\frac{\partial\varphi}{\partial z}$ is the only nonzero component of the field. Therefore:

$$E = -\frac{\partial\varphi}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{2z}{2\sqrt{a^2+z^2}} - 1\right) = \underline{\frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)}.$$

For $z \ll a$, i.e., close to the disk, $\frac{z}{\sqrt{a^2+z^2}} \simeq \frac{z}{a} \ll 1$, and the 2nd term in brackets can be neglected.

Hence, for $z \ll a$ $\quad E \simeq \dfrac{\sigma}{2\varepsilon_0}$,

which is the field of an infinite plane.

For $z \gg a$ we use
$$\frac{1}{\sqrt{a^2+z^2}} = (z^2+a^2)^{-1/2}$$
$$= \frac{1}{z}\left(1+\frac{a^2}{z^2}\right)^{-1/2}$$
$$\simeq \frac{1}{z}\left(1-\frac{a^2}{2z^2}+\dots\right)$$

Substituting this into the expression for E, we have:

$$E \simeq \frac{\sigma}{2\varepsilon_0}\left(1 - z\cdot\frac{1}{z}\left(1-\frac{a^2}{2z^2}+\dots\right)\right)$$
$$= \frac{\sigma}{2\varepsilon_0}\left(1-1+\frac{a^2}{2z^2}\right)$$
$$= \frac{\sigma a^2}{4\varepsilon_0 z^2} = \frac{\sigma \pi a^2}{4\pi\varepsilon_0 z^2} \qquad (\text{for } z \gg a).$$

πa^2 is the total area of the disk and $\sigma \pi a^2 \equiv q$ is the total charge of the disk. Hence, in this limit $E \simeq \dfrac{q}{4\pi\varepsilon_0 z^2}$, as one would have for a point charge. Indeed, when the distance from the disk is much greater than its dimensions, it appears as a point charge.

③ (a) Coulomb's law for the force on charge q at position $\bar{r}$, acted upon by charges q_i at $\bar{r}_i$ $(i=1,2,\dots,N)$ is
$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q q_i(\bar{r}-\bar{r}_i)}{|\bar{r}-\bar{r}_i|^3}.$$

The corresponding electric field $E(\bar{r})$ defined by $\vec{F} = q\,E(\bar{r})$ is given by

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} \frac{q_i\,(\bar{r}-\bar{r}_i)}{|\bar{r}-\bar{r}_i|^3} .$$

(b) <u>Gauss's law</u> states that the flux of the electric field through a closed surface equals the total charge enclosed by the surface divided by ε_0.

Integrating $\bar{E}(\bar{r})$ from part (a) over a closed surface S, we have:

$$\oint_S \bar{E}\cdot d\bar{S} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} q_i \underbrace{\oint_S \frac{(\bar{r}-\bar{r}_i)\cdot d\bar{S}}{|\bar{r}-\bar{r}_i|^3}}$$

This integral gives the solid angle equal to 4π if $\bar{r}_i$ (the apex) lies inside S, and 0 if outside

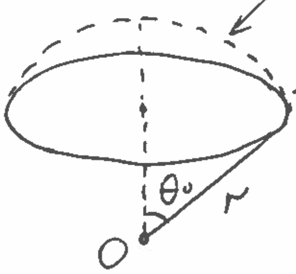
Hence:
$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{\text{over}\\ \text{charges}\\ \text{inside } S}} q_i\, 4\pi$$

$$= \frac{Q}{\varepsilon_0},$$

where $Q = \sum_{\substack{\text{over}\\ \text{charges}\\ \text{inside } S}} q_i$ is the charge enclosed by S.

[Note that $d\vec{S}$ is the outward normal, so the flux <u>out</u> of the closed surface is considered positive.]

(c)



The solid angle sought is given by $\Omega = \frac{A}{r^2}$, where A is the area of the spherical cap and r^2 is the radius of the sphere. To find A, use spherical polar coordinates (r, θ, φ) in which the element of area on the sphere is given by

$$dS = r^2 \sin\theta \, d\theta \, d\varphi .$$

Here θ is the polar angle, i.e., angle with respect to the axis of the system ($0 \le \theta \le \theta_0$) and φ is the azimuthal angle ($0 \le \varphi \le 2\pi$). Hence:

$$A = \int dS = \int_0^{2\pi}\int_0^{\theta_0} r^2 \sin\theta \, d\theta \, d\varphi$$

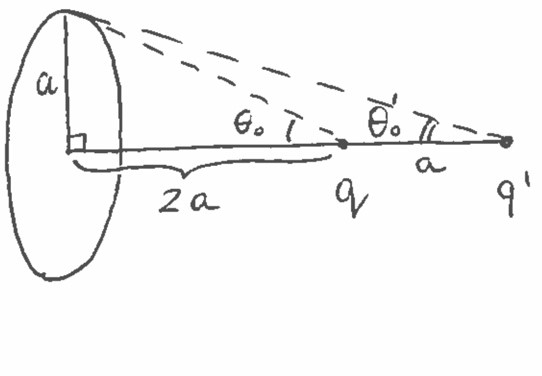
$$= r^2 \int_0^{2\pi} d\varphi \int_0^{\theta_0} \sin\theta \, d\theta$$

$$= r^2 \; 2\pi \; [-\cos\theta]_0^{\theta_0}$$

$$= r^2 \; 2\pi \, [-\cos\theta_0 + \cos 0] = 2\pi r^2 (1 - \cos\theta_0)$$

$$\rightarrow \quad \Omega = \frac{A}{r^2} = 2\pi (1 - \cos\theta_0), \text{ as required.}$$

(d)



The flux through the disc due to q is given by $2\pi(1 - \cos\theta_0)\frac{q}{4\pi\varepsilon_0}$ where

$$\cos\theta_0 = \frac{2a}{\sqrt{4a^2 + a^2}} = \frac{2}{\sqrt{5}} .$$

so the flux is $\frac{q}{2\varepsilon_0}\left(1-\frac{2}{\sqrt{5}}\right)$.

The flux due to q' is

$$2\pi\left(1-\cos\theta_0'\right)\frac{q'}{4\pi\varepsilon_0},$$

where $\cos\theta_0' = \dfrac{3a}{\sqrt{9a^2+a^2}} = \dfrac{3}{\sqrt{10}}$,

so the flux is $\frac{q'}{2\varepsilon_0}\left(1-\frac{3}{\sqrt{10}}\right)$.

Setting the total flux equal to zero,

$$\frac{q}{2\varepsilon_0}\left(1-\frac{2}{\sqrt{5}}\right)+\frac{q'}{2\varepsilon_0}\left(1-\frac{3}{\sqrt{10}}\right)=0$$

gives

$$q' = -q\,\frac{1-\frac{2}{\sqrt{5}}}{1-\frac{3}{\sqrt{10}}}.$$

[Note that the flux of the electric field for a point charge q through a given surface S is

$$\int_S \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\int\frac{\vec{r}\cdot d\vec{S}}{r^3} = \frac{q}{4\pi\varepsilon_0}\,\Omega,$$

where Ω is the solid angle subtended by S, with the apex at the charge (here taken to be at the origin).

4 (a) Applying Gauss's law to the spherically symmetric distribution of charge gives

$$E(r)\,4\pi r^2 = \frac{1}{\varepsilon_0}\int_0^r \rho(r')\,4\pi r'^2\,dr'$$

$$\Rightarrow\quad E(r) = \frac{1}{4\pi\varepsilon_0 r^2}\int_0^r \rho(r')\,4\pi r'^2\,dr',$$

and using $\rho(r') = \frac{q}{4\pi a^2} \frac{e^{-r'/a}}{r'}$, we obtain:

$$E(r) = \frac{1}{4\pi \varepsilon_0 r^2} \int_0^r \frac{q}{\cancel{4\pi} a^2} \frac{e^{-r'/a}}{r'} 4\pi r'^2 dr'$$

$$= \frac{q}{4\pi \varepsilon_0 a^2} \frac{1}{r^2} \int_0^r e^{-r'/a} r' dr'$$ This is to be integrated by parts

$$= \frac{q}{4\pi \varepsilon_0 a^2} \frac{1}{r^2} \left(\left[-a e^{-r'/a} r' \right]_0^r + a \int_0^r e^{-r'/a} dr' \right)$$

$$= \frac{q}{4\pi \varepsilon_0 a^2} \frac{1}{r^2} \left(-a r e^{-r/a} - a^2 \left[e^{-r'/a} \right]_0^r \right)$$

$$= \frac{q}{4\pi \varepsilon_0 a^2} \frac{1}{r^2} \left(-a r e^{-r/a} - a^2 e^{-r/a} + a^2 \right)$$

$$= \frac{q}{4\pi \varepsilon_0} \left(\frac{1}{r^2} - \left(\frac{1}{r^2} + \frac{1}{ar} \right) e^{-r/a} \right).$$

$$\vec{E} = -\vec{\nabla} \varphi = \underbrace{-\frac{\partial \varphi}{\partial r} \vec{e}_r + \dots}$$

only the radial part of the gradient in spherical polar coordinates is needed, since $\vec{E} = E(r) \vec{e}_r$.

Hence $$\frac{d\varphi}{dr} = -\frac{q}{4\pi \varepsilon_0} \left(\frac{1}{r^2} - \left(\frac{1}{r^2} + \frac{1}{ar} \right) e^{-r/a} \right)$$

$$\varphi(r) = -\frac{q}{4\pi \varepsilon_0} \int \left(\frac{1}{r^2} - \left(\frac{1}{r^2} + \frac{1}{ar} \right) e^{-r/a} \right) dr$$

$$= -\frac{q}{4\pi \varepsilon_0} \left[\int \frac{1}{r^2} dr - \int \frac{1}{r^2} e^{-r/a} dr - \frac{1}{a} \int \frac{1}{r} e^{-r/a} dr \right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r} + \underbrace{\int e^{-r/a}\, d\left(\frac{1}{r}\right)}_{\text{This is to be integrated by parts}} - \frac{1}{a}\int \frac{1}{r} e^{-r/a}\, dr\right]$$

$-\frac{1}{r^2}dr = d\left(\frac{1}{r}\right)$ $\qquad$ $\int u\,dv = uv - \int v\,du$

$$= -\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r} + \frac{1}{r}e^{-r/a} - \int \frac{1}{r}\, d\left(e^{-r/a}\right) - \frac{1}{a}\int \frac{1}{r} e^{-r/a}\, dr\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\left(1 - e^{-r/a}\right) + \cancel{\frac{1}{a}\int \frac{1}{r} e^{-r/a}\, dr} - \cancel{\frac{1}{a}\int \frac{1}{r} e^{-r/a}\, dr}\right]$$

$$\Rightarrow \qquad \underline{\varphi(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 - e^{-r/a}\right)}, \quad \text{as required.}$$

(b) Total charge is

$$\int_0^\infty \rho(r)\, 4\pi r^2\, dr = \int_0^\infty \frac{q}{4\pi a^2}\frac{e^{-r/a}}{r}\, 4\pi r^2\, dr$$

$$= \frac{q}{a^2}\int_0^\infty e^{-r/a}\, r\, dr$$

$$= \frac{q}{a^2}\left(\left[-a e^{-r/a} r\right]_0^\infty + a\int_0^\infty e^{-r/a}\, dr\right)$$

$$= \frac{q}{a^2}\left(0 - a^2\left[e^{-r/a}\right]_0^\infty\right)$$

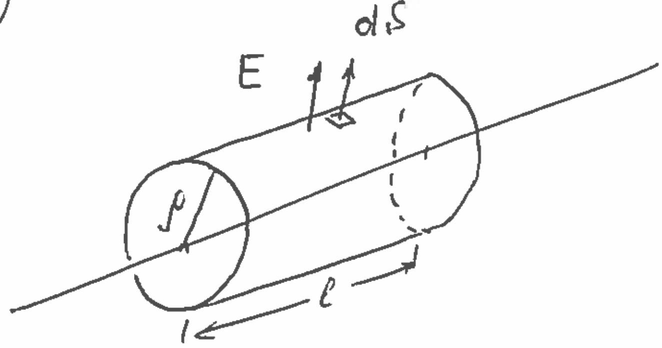
$$= \frac{q}{a^2}\left(-0 + a^2\right) = \underline{q}.$$

Although we have integrated over the whole space, the charge is largely confin[ed] to $r \sim a$ since $e^{-r/a} \to 0$ for $r \gg a$

The potential $\varphi(r)$ at $r \gg a$: $\quad 1 - \underbrace{e^{-r/a}}_{\text{tends to zero for } r\to\infty} \simeq 1$

$$\Rightarrow \quad \varphi(r) = \frac{q}{4\pi\varepsilon_0 r} \quad \text{- potential of point charge } q.$$

(5)



Due to the cylindrical symmetry of the system, the electric field at any point is perpendicular to the line and its magnitude can depend only on the distance from the line.

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

There is no contribution to the flux from the bases, as $\vec{E} \perp d\vec{S}$ there. On the cylindrical surface, on the other hand, $\vec{E} \parallel d\vec{S}$ at any point.

Hence $$\oint \vec{E}\cdot d\vec{S} = \int_{\substack{\text{cyl.}\\ \text{surface}}} E\, dS = E(\rho)\, \underbrace{2\pi\rho\, l}_{\substack{\text{total area}\\ \text{of the cylindrical}\\ \text{surface.}}}$$

$Q = \lambda l$, so we have:

$$E(\rho)\, 2\pi\rho \cancel{l} = \frac{\lambda \cancel{l}}{\varepsilon_0}$$

$$\Longrightarrow \quad E(\rho) = \frac{\lambda}{2\pi\varepsilon_0\rho}.$$

$$\vec{E} = -\vec{\nabla}\varphi = -\Big(\underbrace{\frac{\partial\varphi}{\partial\rho}\vec{e}_\rho + \ldots\ldots}_{\substack{\text{only the } \rho \text{ component of the gradient}\\ \text{in cylindrical coordinates is nonzero.}}}\Big)$$

$$\frac{d\varphi}{d\rho} = -\frac{\lambda}{2\pi\varepsilon_0\rho} \quad \Rightarrow \quad \varphi = -\int \frac{\lambda}{2\pi\varepsilon_0\rho}\, d\rho$$

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\int\frac{d\rho}{\rho} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho + C\text{, as required}$$

For $\rho \to \infty$ $\varphi \to -\infty$. The potential tends to -infinity. Unlike a distribution of charges confined to a finite region in space, there is "no escape" from the infinitely long line and the potential does not tend to a constant value at infinity

(6) For a spherically symmetric system the potential depends only on r. Hence, the Laplace's equation, which it obeys,

$$\nabla^2\varphi = 0\,,$$

in spherical polar coordinates, reads:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = 0$$

$$\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = 0$$

$$r^2\frac{d\varphi}{dr} = C$$

$$\frac{d\varphi}{dr} = \frac{C}{r^2}$$

$$\varphi = \int\frac{C}{r^2}\,dr$$

$$\varphi = -\frac{C}{r} + D\,,$$ where C and D are constants.

The potential between the shells ($a<r<b$) satisfies the boundary conditions $\varphi(a)=\varphi_a$, $\varphi(b)=\varphi_b$. We then have:

$$-\frac{C}{a}+D=\varphi_a \qquad (1)$$

$$-\frac{C}{b}+D=\varphi_b \qquad (2)$$

Subtracting these equations gives

$$-\frac{C}{a}+\frac{C}{b}=\varphi_a-\varphi_b$$

$$C=\frac{\varphi_a-\varphi_b}{\frac{1}{b}-\frac{1}{a}}=\frac{ab(\varphi_a-\varphi_b)}{a-b}.$$

Multiplying (1) by (a), (2) by b and subtracting, gives

$$Da-Db=\varphi_a a-b\varphi_b$$

$$\Rightarrow \quad D=\frac{\varphi_a a-b\varphi_b}{a-b}$$

Hence, the potential is

$$\varphi(r)=-\frac{ab(\varphi_a-\varphi_b)}{a-b}\frac{1}{r}+\frac{\varphi_a a-b\varphi_b}{a-b}$$

or

$$\varphi(r)=\frac{ab(\varphi_a-\varphi_b)}{b-a}\frac{1}{r}+\frac{b\varphi_b-a\varphi_a}{b-a},$$

as required.

For $r>b$: $\quad -\frac{C}{b}+D=\varphi_b$ at $r=b$

For $r\to\infty$ $\varphi\to 0$ so $D=0$ and $C=-b\varphi_b$

$$\Rightarrow \quad \varphi(r)=\frac{b\varphi_b}{r}.$$

7 (a) To determine the behaviour of the potential

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon} \int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV'$$

at large distances $|\vec{r}| \equiv r \gg r'$, we expand:

$$|\vec{r}-\vec{r}\,'|^{-1} = \left[(\vec{r}-\vec{r}\,')^2\right]^{-1/2}$$

$$= \left(\vec{r}^{\,2} - 2\vec{r}\cdot\vec{r}\,' + \vec{r}\,'^2\right)^{-1/2}$$

$$= \left(r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2\right)^{-1/2}$$

$$= \left[r^2\left(1 - \frac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}\right)\right]^{-1/2}$$

$$= \frac{1}{r}\Big(1 \underbrace{- \frac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}}_{\text{small } \alpha \text{ in } (1+\alpha)^{-1/2}}\Big)^{-1/2}$$

Binomial expansion:

$$(1+\alpha)^{-1/2} = 1 - \tfrac{1}{2}\alpha + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}\alpha^2 + \dots$$

$$= 1 - \tfrac{1}{2}\alpha + \tfrac{3}{8}\alpha^2 + \dots$$

$$\Rightarrow |\vec{r}-\vec{r}\,'|^{-1} = \frac{1}{r}\left[1 - \frac{1}{2}\left(-\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}\right) + \frac{3}{8}\left(-\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}\right)^2 + \dots\right]$$

$$= \frac{1}{r}\left[1 + \frac{\vec{r}\cdot\vec{r}\,'}{r^2} - \frac{r'^2}{2r^2} + \frac{3}{8}\left(\frac{4(\vec{r}\cdot\vec{r}\,')^2}{r^4} + \dots\right) + \dots\right]$$

These are terms proportional to $\left(\frac{r'}{r}\right)^3$ and higher, which we do not keep

$$\Rightarrow |\vec{r}-\vec{r}\,'|^{-1} \simeq \frac{1}{r}\left[1+\frac{\vec{r}\cdot\vec{r}\,'}{r^2}-\frac{r'^2}{2r^2}+\frac{3}{2}\frac{(\vec{r}\cdot\vec{r}\,')^2}{r^4}\right] \quad (*)$$

In the last term $\quad \vec{r}\cdot\vec{r}\,' = \sum_{i=1}^{3} x_i x_i'$,

where (x_1, x_2, x_3) are components of $\vec{r}$ (and similarly for $\vec{r}\,' = (x_1', x_2', x_3')$.

$$(\vec{r}\cdot\vec{r}\,')^2 = \sum_{i=1}^{3} x_i x_i' \sum_{j=1}^{3} x_j x_j'$$

} we use different summation indices to avoid confusion

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j\; x_i' x_j' \qquad (**)$$

We can also use

$$\frac{1}{r^2} = \frac{1}{r^4} r^2 = \frac{1}{r^4}\sum_{i=1}^{3} x_i^2 = \frac{1}{r^4}\sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \delta_{ij} \quad (***)$$

to write the 2nd last term in $(*)$ in a form similar to the last one. Substituting $(*)$ with $(**)$ and $(***)$ taken into account into $\varphi(\vec{r})$ gives:

$$\varphi(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_0}\int_V \left[\frac{1}{r}+\frac{\vec{r}\cdot\vec{r}\,'}{r^3}+\frac{1}{2}\frac{1}{r^5}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(3x_ix_j x_i'x_j' - x_i x_j \delta_{ij} r'^2\right)\right] \times \rho(\vec{r}\,')\,dV'$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r}\int_V \rho(\vec{r}\,')\,dV' + \frac{\vec{r}}{r^3}\cdot\int \vec{r}\,'\rho(\vec{r}\,')\,dV' + \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\frac{x_i x_j}{r^5}\int\left(3x_i'x_j' - \delta_{ij} r'^2\right)\rho(\vec{r}\,')\,dV'\right)$$

Introducing $\quad Q = \int \rho(\vec{r}\,')\,dV'$: total charge

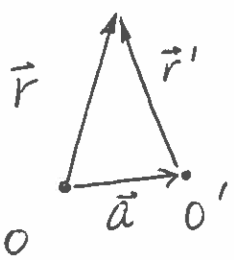
$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,dV'$: dipole moment

and $\quad Q_{ij} = \int (3x_i'x_j' - \delta_{ij} r'^2)\rho(\vec{r}\,')\,dV'$: quadrupole moment tensor

we have:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\Big(\frac{Q}{r} + \frac{\vec{r}\cdot\vec{p}}{r^3} + \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{2}\frac{x_i x_j}{r^5} Q_{ij} + \underbrace{\dots}_{\text{higher-order terms}} \Big)$$

(b)



$\vec{r} = \vec{a} + \vec{r}\,'$,

so the dipole moment with respect to O':

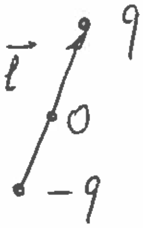
$$\int \vec{r}\,'\rho(\vec{r})\,dV = \int (\vec{r}-\vec{a})\rho(\vec{r})\,dV$$

$$= \underbrace{\int \vec{r}\rho(\vec{r})\,dV}_{\text{dipole moment with respect to O}} - \underbrace{\int \vec{a}\rho(\vec{r})\,dV}_{= \vec{a}\int\rho(\vec{r})\,dV = \vec{a}Q = 0}$$

if $Q = 0$.

$$\Rightarrow \int \vec{r}\,'\rho(\vec{r})\,dV = \int \vec{r}\rho(\vec{r})\,dV.$$

(c) For the charges q and $-q$ separated by $\vec{l}$, choosing the origin at the midpoint, we can write the charge density



$$\rho(\vec{r}) = q\,\delta(\vec{r} - \tfrac{\vec{l}}{2}) - q\,\delta(\vec{r} + \tfrac{\vec{l}}{2})$$

Substituting this into the expression for p:

$$\vec{p} = \int \vec{r}\rho(\vec{r})\,dV = \int \vec{r}\left(q\,\delta(\vec{r}-\tfrac{\vec{l}}{2}) - q\,\delta(\vec{r}+\tfrac{\vec{l}}{2})\right)dV$$

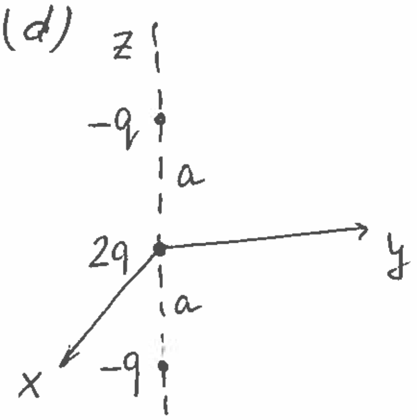
$$= q\Big[\underbrace{\int \vec{r}\,\delta(\vec{r}-\tfrac{\vec{l}}{2})\,dV}_{=\frac{\vec{l}}{2}} - \underbrace{\int \vec{r}\,\delta(\vec{r}+\tfrac{\vec{l}}{2})\,dV}_{=-\frac{\vec{l}}{2}}\Big]$$

($\vec{r} = \frac{\vec{l}}{2}$ is the only point that contributes)

($\vec{r} = -\frac{\vec{l}}{2}$ is the only point that contributes

$$= q\left(\frac{\vec{l}}{2} - \left(-\frac{\vec{l}}{2}\right)\right) = q\vec{l},$$ as required.

(d)



$\left[x \equiv x_1 \quad y \equiv x_2 \quad z \equiv x_3 \right]$

$$Q = -q + 2q - q = \underline{0}.$$

$$\vec{p} = \int \vec{r}\left[-q\,\delta(\vec{r}-\vec{a}) + 2q\,\delta(\vec{r}) - q\,\delta(\vec{r}+\vec{a})\right]dV$$

(where $\vec{a} = (0,0,a)$)

$$= -q\vec{a} + 2q\cdot 0 - q(-\vec{a})$$

$$= -q\vec{a} + q\vec{a} = \underline{0}.$$

$$Q_{ij} = \int (3x_ix_j - \delta_{ij}r^2)\Big[-q\,\delta(\vec{r}-\vec{a}) + \underbrace{2q\,\delta(\vec{r})}\, - q\,\delta(\vec{r}+\vec{a})\Big]dV$$

this term does not contribute since it is only nonzero for $\vec{r}=0$, where $3x_ix_j - \delta_{ij}r^2$ vanishes.

Since the charges are on the z axis, $x_1 = x_2 = 0$ for all of them and any "mixed" component such as Q_{12}, Q_{23} or Q_{13} vanish. ($\delta_{12} = \delta_{23} = \delta_{13} = 0$)

Nonvanishing components are diagonal:

$$Q_{11} \equiv Q_{xx} = \int_V (3x^2 - r^2)[-q\,\delta(\bar{r}-\bar{a}) - q\,\delta(\bar{r}+\bar{a})]\,dV$$

$x = 0$
$r^2 = a^2$
for the points where the charges are

$$= -q\,[(3\cdot 0^2 - a^2) + (3\cdot 0^2 - a^2)]$$

$$= \underline{2qa^2}$$

Similarly,

$$Q_{22} \equiv Q_{yy} = \int_V (3y^2 - r^2)[-q\,\delta(\bar{r}-\bar{a}) - q\,\delta(\bar{r}+\bar{a})]\,dV$$

$$= -q\,[(3\cdot 0^2 - a^2) + (3\cdot 0^2 - a^2)]$$

$$= \underline{2qa^2}.$$

Finally,

$$Q_{33} \equiv Q_{zz} = \int_V (3z^2 - r^2)[-q\,\delta(\bar{r}-\bar{a}) - q\,\delta(\bar{r}+\bar{a})]\,dV$$

$$= -q\,[(3a^2 - a^2) + (3(-a)^2 - a^2)]$$

$$= \underline{-4qa^2}.$$

$$Q_{12} = Q_{13} = Q_{23} = Q_{21} = Q_{31} = Q_{32} = 0.$$

[This quadrupole moment can model the quadrupole moment of the CO_2 molecule $\overset{-q}{O} - \overset{2q}{C} - \overset{-q}{O}$ which is linear and symmetric, with oxygen atoms being negatively charged, and carbon - positively.]