AMA3001 Electromagnetic Theory

Problem sheet 2

SOLUTIONS

① (a) (i) Let us calculate the potential of the polarised dielectric. The dipole moment of the volume element dV' is $\vec{P}(\vec{r}')\,dV'$, where $\vec{r}'$ is the position of dV'. The potential of this dipole at $\vec{r}$ is

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{\vec{P}(\vec{r}')\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,dV' \quad \Big\}\ \text{see Sec. 1.8}$$

Integrating over the volume V of the dielectric, we have:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\vec{P}(\vec{r}')\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,dV'.$$

$$= \frac{1}{4\pi\varepsilon_0}\int_V \vec{P}(\vec{r}')\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\,dV'$$

gradient operator with respect to $\vec{r}'$.

Using $\vec{\nabla}'\cdot(\vec{P}f) = \vec{\nabla}'\cdot\vec{P}\,f + \vec{P}\cdot\vec{\nabla}'f$, where $f = \frac{1}{|\vec{r}-\vec{r}'|}$

we obtain:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \vec{\nabla}'\cdot\left(\vec{P}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\right)dV'$$

$$-\frac{1}{4\pi\varepsilon_0}\int_V \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dV'$$

Applying Gauss's theorem to this term

$$\Rightarrow \varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\vec{P}(\vec{r}')\cdot d\vec{S}'}{|\vec{r}-\vec{r}'|} - \frac{1}{4\pi\varepsilon_0}\int_V \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dV'.$$

The first term has the form of the potential due to surface charges with surface density

$$\sigma_p = \vec{P} \cdot \vec{n}, \quad \text{where } \vec{n} \text{ is the outer normal}$$

to the surface $\left[\vec{P} \cdot d\vec{S} = \underbrace{\vec{P} \cdot \vec{n}}_{\sigma_p} dS \right]$.

The second term is the potential due to the volume distribution of charge with density

$$\rho_p = -\vec{\nabla} \cdot \vec{P}.$$

(ii) By Gauss' law,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} (\rho + \rho_p),$$

where ρ is the density of <u>free</u> charges and ρ_p is the density of polarisation charges.

Hence:

$$\varepsilon_0 \vec{\nabla} \cdot \vec{E} = \rho - \vec{\nabla} \cdot \vec{P}$$

$$\vec{\nabla} \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho,$$

or

$$\underline{\vec{\nabla} \cdot \vec{D} = \rho},$$

where $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ is the electric displacement.

iii. <u>Isotropic</u> means that the polarisation vector $\vec{P}$ is in the same direction as $\vec{E}$, and <u>linear</u> means that $\vec{P}$ is proportional to $\vec{E}$, i.e.

$$\vec{P} = \varepsilon_0 \chi \vec{E},$$

where $\chi = \text{const}$ for a given material, is the electric susceptibility.

(b) i. The system is spherically symmetric, hence, $\vec{E} = E(r)\,\hat{r}$ and $\vec{D} = D(r)\,\hat{r}$. Let the charge of the inner sphere be Q, and outer sphere $-Q$. Applying Gauss's theorem in the integral form

$\oint_S \vec{D}\cdot d\vec{S} = Q_f$ (free charge inside S) to a spherical surface of radius r, $a<r<b$,

we have:

$$4\pi r^2 D(r) = Q$$

$$\Rightarrow \quad D(r) = \frac{Q}{4\pi r^2},$$

and since $\vec{D} = \varepsilon_0 \varkappa \vec{E}$,

$$E(r) = \frac{D(r)}{\varepsilon_0 \varkappa} = \frac{Q}{4\pi\varepsilon_0 \varkappa r^2}.$$

Since $\vec{E} = -\vec{\nabla}\varphi = -\frac{\partial \varphi}{\partial r}\hat{r}$ (for $\varphi = \varphi(r)$),

we have:

$$-\frac{d\varphi}{dr} = \frac{Q}{4\pi\varepsilon_0 \varkappa r^2}$$

$$\varphi = -\int \frac{Q}{4\pi\varepsilon_0\varkappa r^2}\,dr$$

$$\varphi(r) = \frac{Q}{4\pi\varepsilon_0\varkappa r} + \text{Const}.$$

The potential difference between the inner and outer spheres is

$$V = \varphi(a) - \varphi(b) = \frac{Q}{4\pi\varepsilon\varkappa a} + \cancel{\text{const}} - \frac{Q}{4\pi\varepsilon_0\varkappa b} - \cancel{\text{const}}$$

$$\Rightarrow \quad V = \frac{Q}{4\pi\varepsilon_0\varkappa}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon_0\varkappa}\,\frac{b-a}{ab}.$$

Hence, the capacitance $\underline{C = \frac{Q}{V} = 4\pi\varepsilon_0 \frac{\varkappa ab}{b-a}}$.

ii. For $b - a \ll a, b$, $a \approx b$ and $ab \approx a^2$

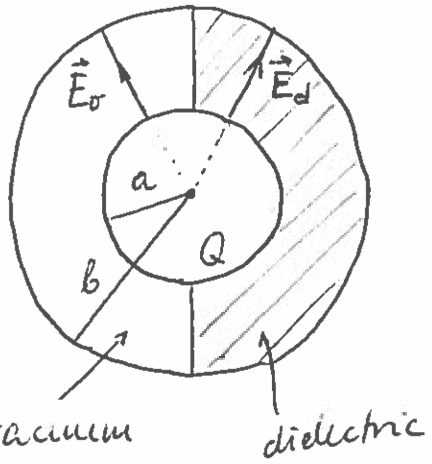
Denoting $b - a \equiv d$, we have:

$$C = 4\pi \varepsilon_0 \frac{\varkappa a^2}{d} = \varepsilon_0 \varkappa \frac{4\pi a^2}{d}$$

$$\Rightarrow \quad C = \varepsilon_0 \varkappa \frac{A}{d},$$

where $A = 4\pi a^2$ is the total area of either of the spherical plates. This formula describes the capacitance of a parallel-plate condenser with plates of area A and spacing d, filled with dielectric of constant $\varkappa$.

2 (a)



[Cross section of the capacitor]

By symmetry, the electric field and displacement $\vec{D}$ are in the radial direction. Denoting the quantities in vacuum by subscript "v" and in dielectric, by "d", we have:

$$\vec{E}_v = E_v(r)\,\hat{r}, \quad \vec{E}_d = E_d(r)\,\hat{r}.$$

From the boundary condition at the interface of two dielectrics, $E_{1t} = E_{2t}$, we have

$$E_v(r) = E_d(r), \qquad (1)$$

since the electric field is parallel to the interface at the interface.

Applying Gauss's law to a sphere of radius r, $a < r < b$, $\oint_S \vec{D} \cdot d\vec{S} = Q$ (charge on inner sphere), The left-hand side is the sum of two contributions, one for each hemisphere:

$$2\pi r^2 D_v(r) + 2\pi r^2 D_d(r) = Q$$

Using $\vec{D} = \varepsilon_0 \chi \vec{E}$, we have:

$$2\pi r^2 \varepsilon_0 E_v(r) + 2\pi r^2 \varepsilon_0 \chi E_d(r) = Q,$$

and using (1),

$$2\pi r^2 \varepsilon_0 (1+\chi) E_v(r) = Q$$

$$\Rightarrow \quad \underline{\underline{E_v(r) = \frac{Q}{2\pi \varepsilon_0 (1+\chi) r^2}}} \quad \left(= E_d(r) \right).$$

(b) From the boundary condition outside the conductor, $D_n = \sigma$, we have:

$$\sigma_v = D_v = \varepsilon_0 E_v(a) = \underline{\underline{\frac{Q}{2\pi \quad (1+\chi) a^2}}},$$

and

$$\sigma_d = D_d = \varepsilon_0 \chi E_d(a) = \underline{\underline{\frac{\chi Q}{2\pi (1+\chi) a^2}}}.$$

[It is easy to check that the total charge on the inner sphere is $\sigma_v \, 2\pi a^2 + \sigma_d \, 2\pi a^2 = Q$.]

(c) From $\sigma_p = \vec{P} \cdot \vec{n}$, and taking into account that the outwards normal $\vec{n}$ on the <u>inner</u> sphere bounding the dielectric is $\vec{n} = -\hat{r}$, we have:

$$\sigma_p = -P(a).$$

From $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$, $\quad P(a) = D_d(a) - \varepsilon_0 E_d(a),$

$$\Rightarrow \quad P(a) = \varepsilon_0 \varkappa E_d(a) - \varepsilon_0 E_d(a)$$

$$= \varepsilon_0 (\varkappa - 1) E_d(a)$$

$$= \frac{(\varkappa - 1) Q}{2\pi (1+\varkappa) a^2}$$

Using the answer from part (a)

$$\Rightarrow \quad \sigma_p = - \frac{(\varkappa - 1) Q}{2\pi (1+\varkappa) a^2} = \underline{\underline{\frac{(1-\varkappa) Q}{2\pi (1+\varkappa) a^2}}} .$$

Note: since $\varkappa > 1$, $\sigma_p < 0$, which makes sense, since the negative polarisation charges form near the positively charged inner sphere.

(d) $\vec{E} = -\vec{\nabla} \varphi$,

and since the electric field is in the radial direction,

$$E_r(r) = - \frac{\partial \varphi}{\partial r} ,$$

so that $\dfrac{d\varphi}{dr} = - \dfrac{Q}{2\pi \varepsilon_0 (1+\varkappa) r^2}$

$$\varphi = - \int \frac{Q}{2\pi \varepsilon_0 (1+\varkappa) r^2} dr$$

$$\varphi(r) = + \frac{Q}{2\pi \varepsilon_0 (1+\varkappa) r} + \text{const}$$

The potential difference:

$$V = \varphi(a) - \varphi(b) = + \frac{Q}{2\pi \varepsilon_0 (1+\varkappa) a} - \frac{Q}{2\pi \varepsilon_0 (1+\varkappa) b}$$

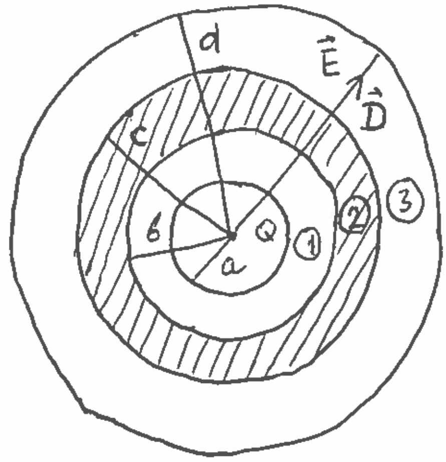
$$= \underline{\frac{Q}{2\pi \varepsilon_0 (1+\varkappa)} \left(\frac{1}{a} - \frac{1}{b} \right)} .$$

(e) The capacitance, $C = \frac{Q}{V}$, is

$$C = \frac{2\pi\varepsilon_0(1+x)}{\frac{1}{a}-\frac{1}{b}} = 2\pi\varepsilon_0(1+x)\,\frac{ab}{b-a}$$

$$= \underbrace{2\pi\varepsilon_0\,\frac{ab}{b-a}}_{\text{This is } \frac{1}{2} \text{ of the capacitence of a capacitor consisting of two sperces of radii } a \text{ and } b \text{ with vacuum inside}} + \underbrace{2\pi\varepsilon_0 x\,\frac{ab}{b-a}}_{\text{This is the capacitance of the same capacitor with the dielectric inside.}} .$$

③ (a) On the cross section of the capacitor shown, we defined regions ① ($a<r<b$), ② ($b<r<c$) and ③ ($c<r<d$). Region ② is the dielectric.



Because of the symmetry of the problem, the electric field and displacement are along the radius:

$\vec{E} = E(r)\,\hat{r}$, $\vec{D} = D(r)\,\hat{r}$.

Using Gauss's law $\oint_S \vec{D}\cdot d\vec{S} = Q_f$ for a sphere of radius r ($a<r<d$), we have

$$4\pi r^2 D(r) = Q,$$

since the charge Q on the inner sphere is the only free charge enclosed by the surface.

$$\Rightarrow \quad D(r) = \frac{Q}{4\pi r^2}.$$

Then using $\vec{D} = \varepsilon_0 \vec{E}$ (regions ① and ③)
and $\vec{D} = \varepsilon \vec{E}$ (region ②), we have:

$$E_1(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{- region 1}$$

$$E_2(r) = \frac{Q}{4\pi\varepsilon r^2} \quad \text{- region 2}$$

$$E_3(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{- region 3}$$

The potential difference between the inner and outer conducting spheres is found from $\vec{E} = -\vec{\nabla}\varphi$, which for $\vec{E} = E(r)\hat{r}$ reads: $E(r) = -\frac{\partial \varphi}{\partial r}$.

$$V = \varphi(a) - \varphi(d) = -\int_a^d d\varphi = +\int_a^d E(r)\,dr$$

$$= \int_a^b E_1(r)\,dr + \int_b^c E_2(r)\,dr + \int_c^d E_3(r)\,dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} + \frac{Q}{4\pi\varepsilon}\int_b^c \frac{dr}{r^2} + \frac{Q}{4\pi\varepsilon_0}\int_c^d \frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{Q}{4\pi\varepsilon}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{c} - \frac{1}{d}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b} + \frac{\varepsilon_0}{\varepsilon}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{c} - \frac{1}{d}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{d} + \left(\frac{\varepsilon_0}{\varepsilon} - 1\right)\frac{1}{b} - \left(\frac{\varepsilon_0}{\varepsilon} - 1\right)\frac{1}{c}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{d} + \frac{\varepsilon_0 - \varepsilon}{\varepsilon}\left(\frac{1}{b} - \frac{1}{c}\right)\right]$$

Multiplying by $\frac{4\pi\varepsilon_0}{Q}$, we have:

$$\underline{\underline{4\pi\varepsilon_0 \frac{V}{Q} = \frac{4\pi\varepsilon_0}{C} = \frac{1}{a} - \frac{1}{d} + \frac{\varepsilon_0 - \varepsilon}{\varepsilon}\left(\frac{1}{b} - \frac{1}{c}\right)}}, \text{ as required.}$$

(b) From $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$, and taking into account the (radial) direction of $\vec{D}$ and $\vec{E}$ vectors,

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = D(r)\,\hat{r} - \varepsilon_0 E_2(r)\,\hat{r}$$

$E_2(r)$ is inside the dielectric

$$= \left(\frac{Q}{4\pi r^2} - \varepsilon_0 \frac{Q}{4\pi \varepsilon r^2} \right) \hat{r}$$

$$= \underline{\underline{\frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon} \frac{Q}{r^2}\,\hat{r}}}, \text{ as required.}$$

(c) Polarisation charges: $\rho_p = -\vec{\nabla}\cdot\vec{P}$ (volume)

$\sigma_p = \vec{P}\cdot\vec{n}$ (where $\vec{n}$ is normal outer with respect to the dielectric).

The $\underline{\text{radial}}$ component of the divergence gives:

$$\rho_p = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P(r)\right)$$

Here we use $\vec{P} = P(r)\,\hat{r}$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon}\frac{Q}{r^2}\right)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon} Q\right) = \underline{0}.$$

The surface charge density on the surface at $r = b$ is obtained using $\vec{n} = -\hat{r}$:

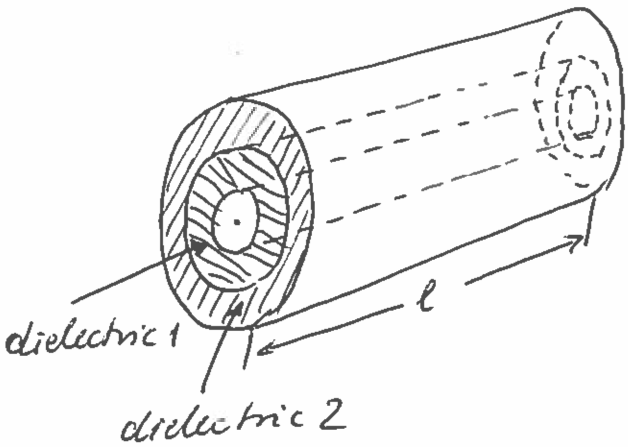
$$\sigma_p = \frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon}\frac{Q}{b^2}\,\hat{r}\cdot(-\hat{r}) = \underline{\underline{-\frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon}\frac{Q}{b^2}}}.$$

On the dielectric surface $r = c$, $\vec{n} = \hat{r}$, so that

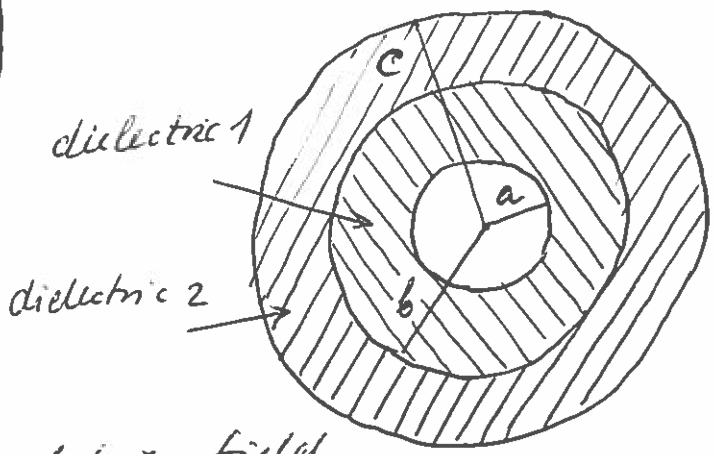
$$\sigma_p = \frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon}\frac{Q}{c^2}\,\hat{r}\cdot\hat{r} = \underline{\underline{\frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon}\frac{Q}{c^2}}}.$$

$\underline{\text{Note}}$: total polarisation charge: $-\frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon}\frac{Q}{b^2}4\pi b^2 + \frac{\varepsilon - \varepsilon_0}{4\pi\varepsilon}\frac{Q}{c^2}4\pi c^2 = 0$

(4) The diagram below shows a section of the cable of length l.



Cross sectional view:



By symmetry, the electric field and displacement are in the radial direction, perpendicular to the axis of the cylinders:

$$\vec{D} = D(\rho)\,\hat{\rho} \quad , \quad \vec{E} = E(\rho)\,\hat{\rho} \, , \tag{1}$$

where ρ is the distance from the axis. [We use cylindrical coordinates here with the z axis being the axis of the cylinders.]

Assuming that the length l of the inner cylinder carries charge Q we apply Gauss's law

$\oint_S \vec{D}\cdot d\vec{S} = Q$, taking S to be a cylinder of radius ρ and length l, coaxial to the other cylinders. Since $\vec{D}$ is $\perp$ to the axis, there is no flux through the bases of the cylinder, so we have:

$$D(\rho)\,\underbrace{2\pi\rho l}_{\text{area of the cylindrical surface}} = Q$$

$$\Rightarrow \qquad D(\rho) = \frac{Q}{2\pi\rho l} \tag{2}$$

From $\vec{D} = \varepsilon_0 \varkappa \vec{E}$ we have inside dielectric

1:
$$E_1(\rho) = \frac{D(\rho)}{\varepsilon_0 \varkappa_1} = \frac{Q}{2\pi \varepsilon_0 \varkappa_1 \rho l}, \qquad (3)$$

and inside dielectric 2:

$$E_2(\rho) = \frac{D(\rho)}{\varepsilon_0 \varkappa_2} = \frac{Q}{2\pi \varepsilon_0 \varkappa_2 \rho l}. \qquad (4)$$

Using these we can find the potential difference:

$$V = \int_a^c \vec{E} \cdot d\vec{\rho} = \int_a^b E_1(\rho) d\rho + \int_b^c E_2(\rho) d\rho$$

$$= \frac{Q}{2\pi \varepsilon_0 l} \left[\frac{1}{\varkappa_1} \int_a^b \frac{d\rho}{\rho} + \frac{1}{\varkappa_2} \int_b^c \frac{d\rho}{\rho} \right]$$

$$= \frac{Q}{2\pi \varepsilon_0 l} \left[\frac{1}{\varkappa_1} (\ln b - \ln a) + \frac{1}{\varkappa_2} (\ln c - \ln b) \right]$$

$$= \frac{Q}{2\pi \varepsilon_0 l} \left(\frac{1}{\varkappa_1} \ln \frac{b}{a} + \frac{1}{\varkappa_2} \ln \frac{c}{b} \right)$$

This gives:
$$\frac{Q}{2\pi \varepsilon_0 l} = \frac{V}{\frac{1}{\varkappa_1} \ln \frac{b}{a} + \frac{1}{\varkappa_2} \ln \frac{c}{b}},$$

so that
$$\vec{D} = \frac{\varepsilon_0 V}{\frac{1}{\varkappa_1} \ln \frac{b}{a} + \frac{1}{\varkappa_2} \ln \frac{c}{b}} \cdot \frac{\hat{\rho}}{\rho} \qquad \text{(from (1) and (2))}$$

$$\vec{E}_1 = \frac{V}{\frac{1}{\varkappa_1} \ln \frac{b}{a} + \frac{1}{\varkappa_2} \ln \frac{c}{b}} \frac{\hat{\rho}}{\varkappa_1 \rho} \qquad \text{(from (1) and (3))}$$

$$\vec{E}_2 = \frac{V}{\frac{1}{\varkappa_1} \ln \frac{b}{a} + \frac{1}{\varkappa_2} \ln \frac{c}{b}} \frac{\hat{\rho}}{\varkappa_2 \rho} \qquad \text{(from (1) and (4))}.$$

Using $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ we find

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E}.$$

In dielectric 1:

$$\vec{P}_1 = \frac{\varepsilon_0 V}{\frac{1}{\varkappa_1}\ln\frac{b}{a} + \frac{1}{\varkappa_2}\ln\frac{c}{b}}\left(1 - \frac{1}{\varkappa_1}\right)\frac{\hat{\rho}}{\rho}$$

$$= \frac{\varepsilon_0(\varkappa_1 - 1)V}{\ln\frac{b}{a} + \frac{\varkappa_1}{\varkappa_2}\ln\frac{c}{b}}\frac{\hat{\rho}}{\rho}$$

$$= \underline{\underline{\frac{\varepsilon_0(\varkappa_1 - 1)\varkappa_2 V}{\varkappa_2\ln\frac{b}{a} + \varkappa_1\ln\frac{c}{b}}\frac{\hat{\rho}}{\rho}}}\,, \quad \text{as required.}$$

In dielectric 2:

$$\vec{P}_2 = \frac{\varepsilon_0 V}{\frac{1}{\varkappa_1}\ln\frac{b}{a} + \frac{1}{\varkappa_2}\ln\frac{c}{b}}\left(1 - \frac{1}{\varkappa_2}\right)\frac{\hat{\rho}}{\rho}$$

$$= \underline{\underline{\frac{\varepsilon_0(\varkappa_2 - 1)\varkappa_1 V}{\varkappa_2\ln\frac{b}{a} + \varkappa_1\ln\frac{c}{b}}\frac{\hat{\rho}}{\rho}}}\,, \quad \text{as required.}$$

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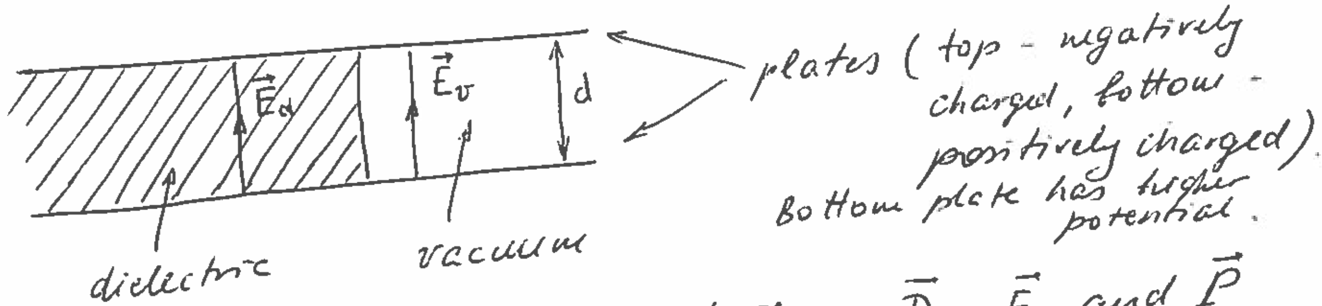

By symmetry, all three vectors, $\vec{D}$, $\vec{E}$ and $\vec{P}$ are perpendicular to the plates everywhere. [Here we neglect the edge effects since d is small compared with the dimensions of the plates.]

Between the plates $\quad \vec{\nabla}\cdot\vec{D} = 0 \quad \Rightarrow \quad \underline{\vec{D} = \text{const}}$.

$$E_v = \frac{D_v}{\varepsilon_0}\,, \qquad E_d = \frac{D_d}{\varepsilon_0 \varkappa}.$$

Considering the boundary dielectric / vacuum, from $E_{1t} - E_{2t} = 0$ we deduce that $E_v = E_d$.

This gives the potential difference $\Delta\varphi = \int \bar{E} \cdot d\bar{r}$

as $\Delta\varphi = E_v d = E_d d$.

$$\Rightarrow \quad \underline{\underline{E_v = E_d = \frac{\Delta\varphi}{d}}} \; .$$

(b) $$D_v = \varepsilon_0 E_v = \frac{\varepsilon_0 \Delta\varphi}{d}$$

$$D_d = \varepsilon_0 \varkappa E_d = \frac{\varepsilon_0 \varkappa \Delta\varphi}{d} .$$

From $D_n = \sigma$, on the bottom plate in contact with the dielectric, the surface charge density is

$$\sigma_d = D_d = \underline{\underline{\frac{\varepsilon_0 \varkappa \Delta\varphi}{d}}} ,$$

and the part of the plate exposed to vacuum,

$$\sigma_v = D_v = \underline{\underline{\frac{\varepsilon_0 \Delta\varphi}{d}}} .$$

(c) The polarisation surface charge density $\sigma_p = \bar{P} \cdot \bar{n}$, where $\bar{n}$ is the outer normal to the dielectric surface.

From $\bar{D} = \varepsilon_0 \bar{E} + \bar{P}$, $\bar{P} = \bar{D} - \varepsilon_0 \bar{E}$,

so $$P = D_d - \varepsilon_0 E_d = \frac{\varepsilon_0 \varkappa \Delta\varphi}{d} - \frac{\varepsilon_0 \Delta\varphi}{d} = \frac{\varepsilon_0 (\varkappa - 1) \Delta\varphi}{d} .$$

Since the outer normal $\bar{n}$ is downwards on the diagram (near the bottom plate), $\underline{\underline{\sigma_p = -\frac{\varepsilon_0 (\varkappa - 1)\Delta\varphi}{d}}}$.

(d) $$\sigma_d + \sigma_p = \frac{\varepsilon_0 \varkappa \Delta\varphi}{d} - \frac{\varepsilon_0 (\varkappa - 1) \Delta\varphi}{d} = \frac{\varepsilon_0 \Delta\varphi}{d} = \sigma_v ,$$

as required.