AMA3001 Electromagnetic Theory

Problem sheet 3

SOLUTIONS

① (a) Suppose, there are two potentials, ϕ_1 and ϕ_2, that satisfy the equation $\vec{\nabla}\cdot\vec{D} = \rho$, where ρ is the volume density of free charges and $\vec{D} = \varepsilon \vec{E} = -\varepsilon \vec{\nabla}\phi$, i.e.,

$$\vec{\nabla}\cdot(\varepsilon\vec{\nabla}\phi_1) = -\rho = \vec{\nabla}\cdot(\varepsilon\vec{\nabla}\phi_2).$$

Let ϕ_1 and ϕ_2 also satisfy the boundary conditions, i.e., given potentials on one set of conductors and given total charges on another. [Recall: the total charge on a conductor can be found as an integral over its surface, $-\oint_S \varepsilon \frac{\partial \varphi}{\partial n} dS$.]

Consider a function $\phi = \phi_1 - \phi_2$.

It satisfies the equation

$$\vec{\nabla}\cdot(\varepsilon\vec{\nabla}\phi) = 0,$$

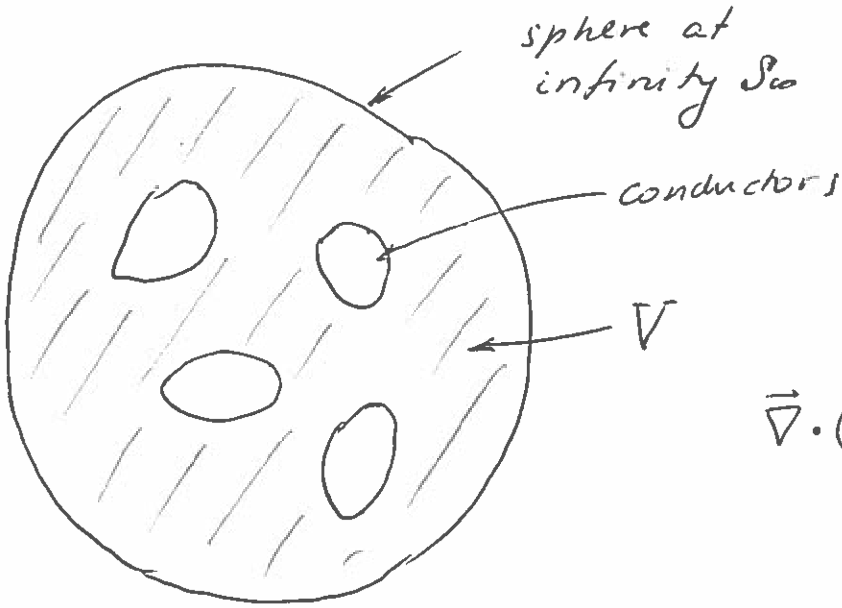
and takes zero values on all conductors with given potentials and $-\oint_S \varepsilon \frac{\partial\phi}{\partial n} dS = 0$ for any conductor with a given charge.

We will show that $\phi = 0$ everywhere, so that $\phi_1 = \phi_2$

Consider the integral

$$\int_V \vec{\nabla}\phi\cdot\varepsilon\vec{\nabla}\phi \, dV \qquad (1)$$

over the volume outside all the conductors and enclosed in a sphere of an arbitrarily large radius (so-called, "sphere at infinity" S_∞).



Using the identity

$$\vec{\nabla}\cdot(\phi\,\varepsilon\vec{\nabla}\phi) = \vec{\nabla}\phi\cdot\varepsilon\vec{\nabla}\phi + \phi\,\underbrace{\vec{\nabla}\cdot(\varepsilon\vec{\nabla}\phi)}_{=0}$$

we transform integral (1) into

$$\int_V \vec{\nabla}\cdot(\phi\,\varepsilon\vec{\nabla}\phi)\,dV \underset{\substack{\uparrow \\ \text{by Gauss's} \\ \text{theorem}}}{=} \oint_{S_\infty}\phi\,\varepsilon\vec{\nabla}\phi\cdot d\vec{S} + \oint_{S_c}\phi\,\varepsilon\vec{\nabla}\phi\cdot d\vec{S},$$

where S_c is the surface of all conductors, with $d\vec{S}$ directed inwards (i.e., outside of V). On the surface S_i of each conductor, we have:

$$\oint_{S_i}\phi\,\varepsilon\vec{\nabla}\phi\cdot d\vec{S} = \phi_i\oint\varepsilon\vec{\nabla}\phi\cdot(-\vec{n}\,dS),$$

[since $\phi_i = \text{const}$ on the surface of the conductor]

$\vec{n}$ above is the <u>outer</u> normal for the conductor $\Rightarrow$ $\vec{\nabla}\phi\cdot\vec{n} = \frac{\partial\phi}{\partial n}$.

Hence

$$\oint_{S_i}\phi\,\varepsilon\vec{\nabla}\phi\cdot d\vec{S} = \phi_i\left(-\oint_{S_i}\varepsilon\frac{\partial\phi}{\partial n}\,dS\right),$$

which is <u>zero</u>, as either $\phi_i = 0$ or the integral in brackets vanishes.

The contribution of the sphere at infinity,

$$\oint_{S_\infty} \phi\, \varepsilon \vec{\nabla}\phi \cdot d\vec{S}, \quad \text{also vanishes, since}$$

$\phi \sim \frac{1}{r}$ at large r, and $\vec{\nabla}\phi \sim \frac{1}{r^2}$,

so that $\phi\vec{\nabla}\phi \sim \frac{1}{r^3}$, while the area of the sphere increases as $\sim r^2$, so $\frac{1}{r^3} r^2 \xrightarrow[r\to\infty]{} 0$.

Therefore

$$\int_V \vec{\nabla}\phi \cdot \varepsilon \vec{\nabla}\phi \, dV = \int_V \varepsilon (\vec{\nabla}\phi)^2 dV = 0.$$

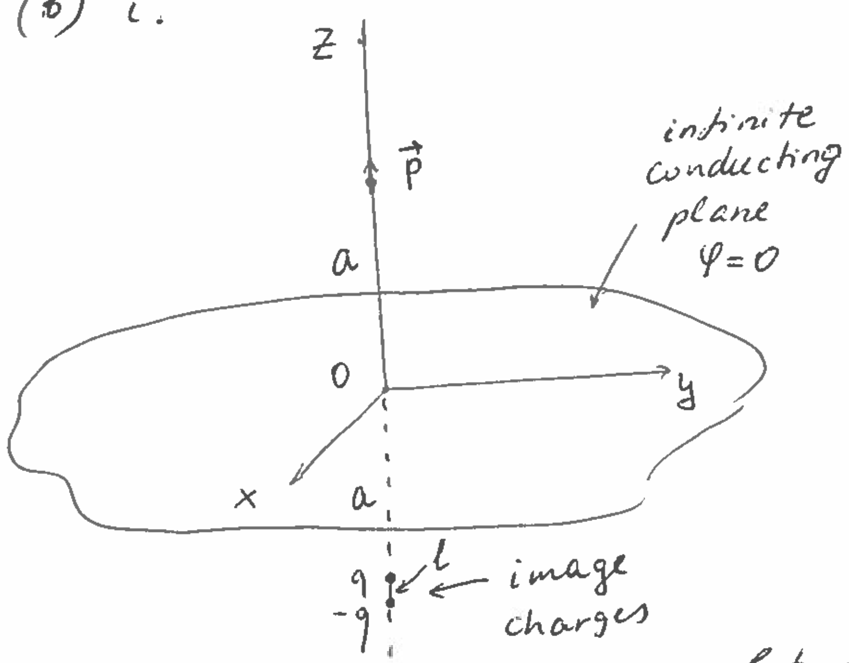
Since $\varepsilon > 0$ everywhere, we must have $\vec{\nabla}\phi = 0$.

This means that $\phi = \text{const}$. However, $\phi \to 0$ at $r \to \infty$, so $\phi = 0$ everywhere. Hence,

$$\underline{\phi_1 = \phi_2},$$

and the solution of the electrostatic problem is unique.

(b) i.



The dipole can be considered as two charges, q and $-q$ at a small distance l from each other $l \Big[\begin{smallmatrix} q \\ -l \end{smallmatrix}$ $(p = ql)$

Its image is then given by charges $-q$ and q placed symmetrically below the plane, near $z = -a$. The distance between the image charges is l, so the image is also a dipole $\vec{p}$.

The potential of a dipole at the origin is $\varphi = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$.

We have one dipole at $\vec{r}_1 = (0,0,a)$ and the other (image) at $\vec{r}_2 = (0,0,-a)$.

The total potential above the plane is given by the sum of the potentials of the real and image dipoles:

$$\varphi(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}\cdot(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} + \frac{\vec{p}\cdot(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}\right],$$

where $\vec{r} = (x,y,z)$, so that

$$\vec{r}-\vec{r}_1 = (x,y,z-a), \qquad \vec{r}-\vec{r}_2 = (x,y,z+a),$$

and using $\vec{p} = (0,0,p)$, we obtain:

$$\varphi(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{p(z-a)}{\left[x^2+y^2+(z-a)^2\right]^{3/2}} + \frac{p(z+a)}{\left[x^2+y^2+(z+a)^2\right]^{3/2}}\right].$$

We can verify that on the plane $(z=0)$

$$\varphi(x,y,0) = 0, \quad \text{as required.}$$

ii. The surface charge density on the conducting plane is found from the relation $E_n = \frac{\sigma}{\varepsilon_0}$, so that

$$\sigma = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial\varphi}{\partial n}.$$

In our case the direction normal to the surface is z, so that

$$\sigma(x,y) = -\varepsilon_0 \left.\frac{\partial\varphi}{\partial z}\right|_{z=0}.$$

Using the quotient and chain rules to evaluate the derivatives we have:

$$\frac{\partial \varphi}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{ \frac{p\,[x^2+y^2+(z-a)^2]^{3/2} - p(z-a)\frac{3}{2}[x^2+y^2+(z-a)^2]^{1/2}\,2(z-a)}{[x^2+y^2+(z-a)^2]^3} \right.$$

$$\left. + \frac{p\,[x^2+y^2+(z+a)^2]^{3/2} - p(z+a)\frac{3}{2}[x^2+y^2+(z+a)^2]^{1/2}\,2(z+a)}{[x^2+y^2+(z+a)^2]^3} \right\}$$

Setting $z=0$, we obtain:

$$\sigma(x,y) = -\varepsilon_0 \frac{1}{4\pi\varepsilon_0}\left\{ \frac{p(x^2+y^2+a^2)+3pa(-a)}{(x^2+y^2+a^2)^{5/2}} + p\frac{(x^2+y^2+a^2)-3pa^2}{(x^2+y^2+a^2)^{5/2}} \right\}$$

$$= -\frac{1}{4\pi}\,\frac{2p(x^2+y^2+a^2-3a^2)}{(x^2+y^2+a^2)^{5/2}}$$

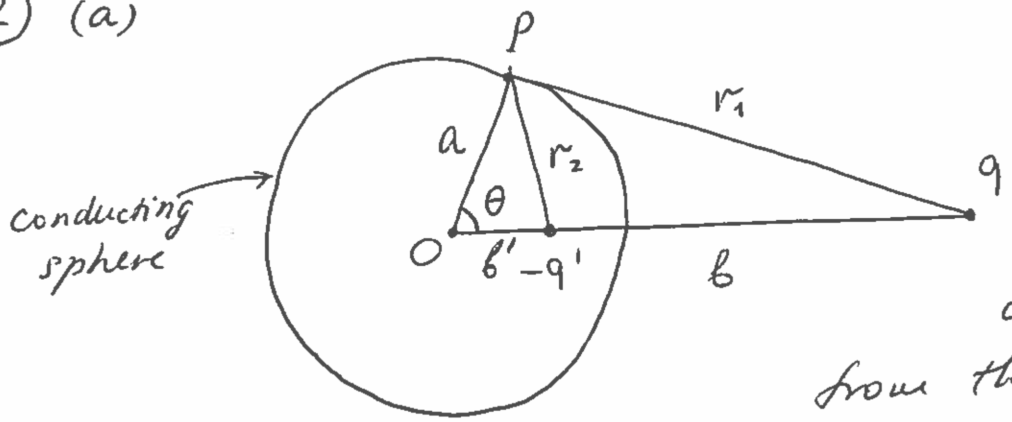
$$= -\frac{p}{2\pi}\,\frac{x^2+y^2-2a^2}{(x^2+y^2+a^2)^{5/2}}.$$

Note that just below the dipole (at $(x,y)=(0,0)$) $\sigma>0$, as the negative end of the dipole is closer to the plane than its positive end. For $x^2+y^2>2a^2$, however, $\sigma<0$.

One can check that the total charge on the plane is zero (i.e. $\int \sigma\, dx\,dy = -\frac{p}{2\pi}\iint_{-\infty}^{+\infty} \frac{x^2+y^2-2a^2}{(x^2+y^2+a^2)^{5/2}}\,dx\,dy = 0$,

[The integral can be done in plane polar coordinates; $dx\,dy \to \rho\,d\rho$

(2) (a)



Let us place the image charge $-q'$ on the line Oq, at a distance b' from the centre of the sphere.

The potential at point P on the sphere is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r_1} - \frac{q'}{r_2}\right], \quad \text{where}$$

$$r_1 = \sqrt{a^2+b^2-2ab\cos\theta} \quad \text{and} \quad r_2 = \sqrt{a^2+b'^2-2ab'\cos\theta}.$$

Setting $\varphi = 0$ on the sphere require

$$\frac{q}{\sqrt{a^2+b^2-2ab\cos\theta}} = \frac{q'}{\sqrt{a^2+b'^2-2ab'\cos\theta}},$$

which gives:

$$\frac{a^2}{q^2} + \frac{b^2}{q^2} - \frac{2ab}{q^2}\cos\theta = \frac{a^2}{q'^2} + \frac{b'^2}{q'^2} - \frac{2ab'}{q'^2}\cos\theta$$

For this to hold for all θ, we need

$$\frac{a^2}{q^2} + \frac{b^2}{q^2} = \frac{a^2}{q'^2} + \frac{b'^2}{q'^2} \quad \text{and} \quad \frac{2ab}{q^2} = \frac{2ab'}{q'^2}$$

From the latter, $\quad q'^2 = q^2\dfrac{b'}{b}. \quad (1)$

Substituting into the first equation gives:

$$\frac{a^2}{q^2} + \frac{b^2}{q^2} = \frac{a^2 b}{q^2 b'} + \frac{b'^2 b}{q^2 b'}.$$

Multiplying by q^2 and by b' we obtain:

$$b b'^2 - (a^2+b^2)b' + a^2 b = 0 .$$

$$\Rightarrow \quad b' = \frac{a^2+b^2 \pm \sqrt{(a^2+b^2)^2 - 4a^2b^2}}{2b}$$

$$= \frac{a^2+b^2 \pm \sqrt{(b^2-a^2)^2}}{2b}$$

$$= \frac{a^2+b^2 \pm (b^2-a^2)}{2b}$$

Using "+" sign, we obtain $b' = b$, which is trivial.
Using "−" sign, we find

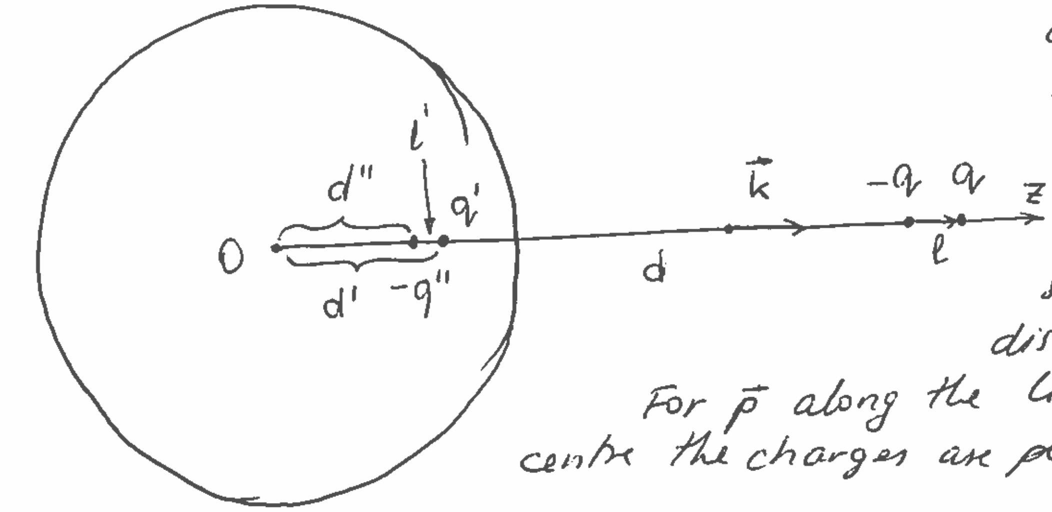
$$\underline{\underline{b' = \frac{a^2}{b}}} . \qquad (2)$$

Note that $b' = a\frac{a}{b} < a$, so the image charge is inside the sphere, as expected.
Its magnitude is found from (1):

$$q'^2 = q^2 \frac{a^2}{b^2}$$

$$\Rightarrow \quad \underline{\underline{q' = \frac{qa}{b}}} . \qquad (3)$$

(b)



i. The point dipole can be considered as a system of two charges, q and $-q$, separated by distance l, $p = ql$.
For $\vec{p}$ along the line from the centre the charges are placed as shown.

Let $-q$ be at a distance d from O, and q at a distance $d+l$ from O. The image q' of $-q$ is then

$$q' = \frac{qa}{d}$$

$$\text{at } d' = \frac{a^2}{d} \text{ from } O.$$

The image of q is $-q''$, with

$$q'' = \frac{qa}{d+l}$$

$$\text{at } d'' = \frac{a^2}{d+l}$$

Regarding l as small, $l \ll d$, we can expand

$$q'' = \frac{qa}{d(1+l/d)} = \frac{qa}{d}(1+l/d)^{-1} \simeq \frac{qa}{d}\left(1-\frac{l}{d}\right)$$

(using binomial expansion* to first order)

$$\Rightarrow \quad -q'' = -\frac{qa}{d} + \frac{qal}{d^2} = -\frac{qa}{d} + \frac{pa}{d^2}.$$

The distance l' between the positive and negative image charges is

$$l' = d' - d'' = \frac{a^2}{d} - \frac{a^2}{d+l} = \frac{a^2}{d} - \frac{a^2}{d(1+l/d)}$$

$$= \frac{a^2}{d}\left[1-(1+l/d)^{-1}\right] \simeq \frac{a^2}{d}\left[1-1+\frac{l}{d}\right]$$

$$\Rightarrow \quad l' = \frac{a^2 l}{d^2}.$$

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* Binomial expansion: $(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots$

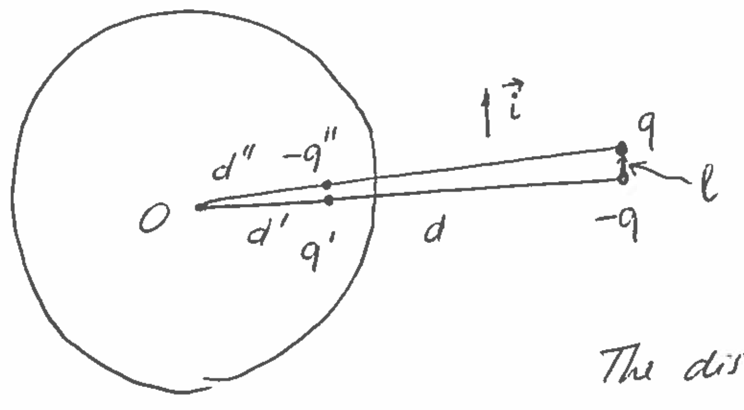
Thus we see that the image consists of two charges, $\frac{qa}{d}$ and $-\frac{qa}{d}+\frac{pa}{d^2}$, separated by distance $l'=\frac{a^2 l}{d^2}$. In the limit $l\to 0$, this becomes a dipole at $\underline{\underline{d'=\frac{a^2}{d}}}$, of magnitude

$$p'=\frac{qa}{d}l'=\frac{qa}{d}\frac{a^2 l}{d^2}=\frac{a^3 ql}{d^3}=\underline{\underline{\frac{pa^3}{d^3}}},$$

and a point charge $\underline{\underline{\frac{pa}{d^2}}}$ at the same point. Taking into account the direction of the dipole, we can write

$$\underline{\underline{\vec{p}\,'=\frac{pa^3}{d^3}\vec{k}}}, \text{ or } \underline{\underline{\vec{p}\,'=\vec{p}\frac{a^3}{d^3}}}.$$

ii. If the dipole is perpendicular to the line through the centre, we have the following picture:



If the distance from $-q$ to the centre is d, its image

$$q'=\frac{qa}{d}$$

at $d'=\frac{a^2}{d}$.

The distance between q and O is $\sqrt{d^2+l^2}$, so its image $-q''$ is at

$$d''=\frac{a^2}{\sqrt{d^2+l^2}} \quad\text{and}\quad q''=\frac{qa}{\sqrt{d^2+l^2}}.$$

Taking the limit $l\to 0$ $\left(\frac{l}{d}\ll 1\right)$, we see that $d''\simeq d'$ and $q''\simeq q'$ up to 1st-order corrections in l/d (since $(d^2+l^2)^{-1/2}=d\left(1+\frac{l^2}{d^2}\right)^{-1/2}\simeq d\left(1-\frac{1}{2}\frac{l^2}{d^2}\right)$).

Thus we see that the image is a pure dipole with charges $q' = \frac{qa}{d}$ and $-q' = -\frac{qa}{d}$ at $d' = \frac{a^2}{d}$, and directed <u>downwards</u>. The distance between q' and $-q''$ $(\approx -q')$ is found from similar triangles as

$$\frac{l'}{l} = \frac{d'}{d} \Rightarrow l' = \frac{ld'}{d} = \frac{la^2}{d^2}$$

Hence, the magnitude of the image dipole is

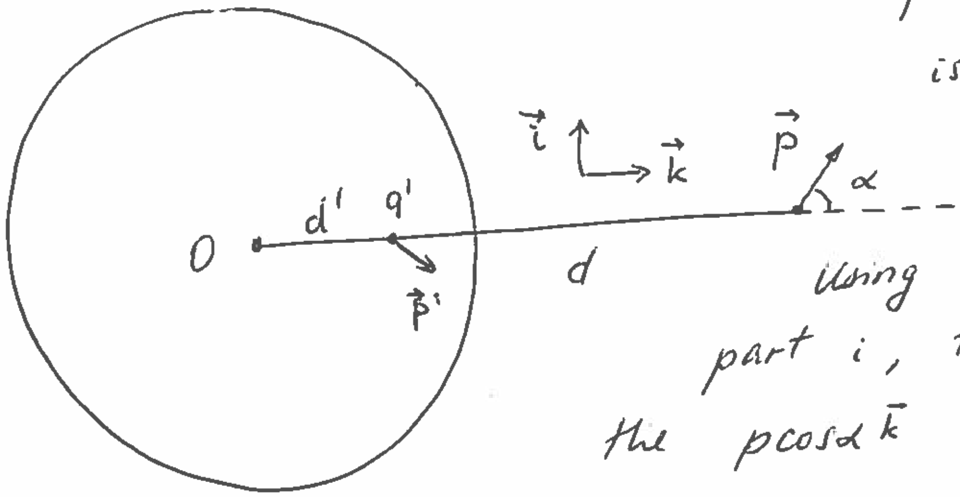
$$p' = q'l' = \frac{qa}{d} \cdot \frac{la^2}{d^2} = \frac{qla^3}{d^3} = \underline{\underline{p\frac{a^3}{d^3}}}$$

Taking into account their directions, $\vec{p} = p\vec{i}$,

and $$\underline{\underline{\vec{p}\,' = -p\frac{a^3}{d^3}\vec{i} = -\vec{p}\frac{a^3}{d^3}}}.$$

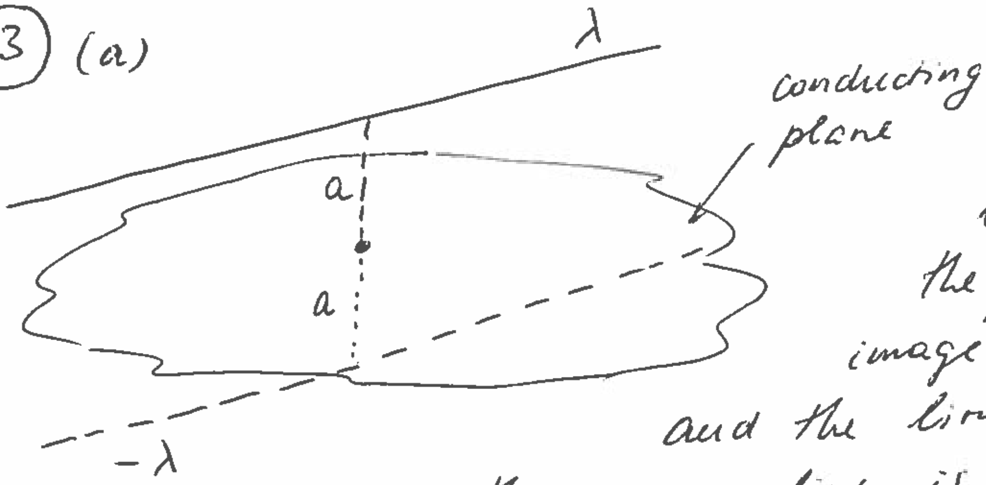
iii.

The dipole moment $\vec{p} = p\sin\alpha\,\vec{i} + p\cos\alpha\,\vec{k}$ is a superposition of two dipoles.



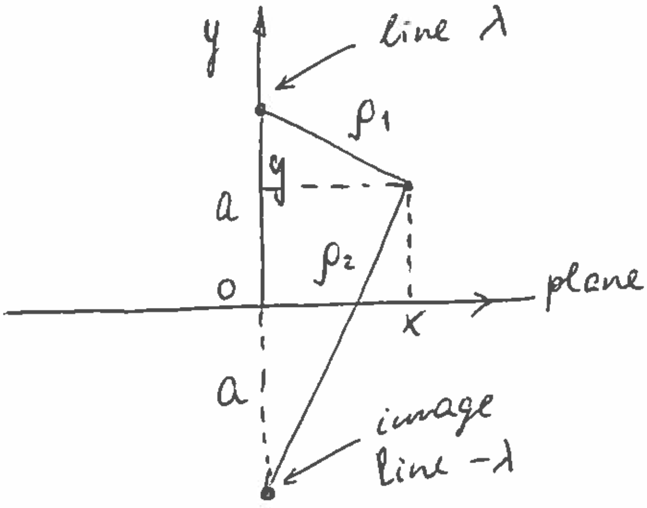
Using the result from part i, the image of the $p\cos\alpha\,\vec{k}$ dipole at d is the dipole $\frac{p\cos\alpha\, a^3}{d^3}\vec{k}$ and point charge $\frac{p\cos\alpha\, a}{d^2}$, placed at a distance $d' = \frac{a^2}{d}$ from the centre of the sphere. The image of $p\sin\alpha\,\vec{i}$ is, by the result from ii, the dipole $-\frac{p\sin\alpha\, a^3}{d^3}\vec{i}$, placed at the same point. Hence, the total image system consists of the dipole $\vec{p}\,' = (-p\sin\alpha\,\vec{i} + p\cos\alpha\,\vec{k})\dfrac{a^3}{d^3}$ and point charge $q' = p\dfrac{\cos\alpha\, a}{d^2}$, placed at $d' = \dfrac{a^2}{d}$.

3 (a)



If an infinite uniformly charged line is placed below the plane as a mirror image of the original line, and the linear charge density on the image line is $-\lambda$, the sum of the potentials of the real line and image line will be zero at any point on the plane.

View along the lines and plane:



The potential of a uniformly charged line is $\varphi = -\dfrac{\lambda}{2\pi\varepsilon_0}\ln\rho$, where ρ is the distance from the line.

Hence, for any point (x, y) in the plane perpendicular to the lines the potential is

$$\varphi(x,y) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho_1 - \frac{-\lambda}{2\pi\varepsilon_0}\ln\rho_2$$

With the x and y coordinates chosen as shown on the diagram, $\rho_1 = \sqrt{x^2 + (y-a)^2}$, $\rho_2 = \sqrt{x^2 + (y+a)^2}$.

Hence:

$$\varphi(x,y) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{x^2+(y-a)^2} + \frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{x^2+(y+a)^2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left[x^2+(y+a)^2\right] - \ln\left[x^2+(y-a)^2\right]\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{x^2+(y+a)^2}{x^2+(y-a)^2}.$$

It is easy to see that for $y=0$ $\varphi(x,0) = \frac{\lambda}{4\pi\varepsilon_0}\ln 1 = 0$, as required.

The surface charge density on the plane is found from $\sigma = -\varepsilon_0 \frac{\partial\varphi}{\partial n}$, where the derivative is with respect to the outer normal to the conducting surface.

In our case,

$$\sigma(x) = -\varepsilon_0 \left.\frac{\partial\varphi}{\partial y}\right|_{y=0}$$

$$= -\frac{\lambda}{4\pi}\left[\frac{2(y+a)}{x^2+(y+a)^2} - \frac{2(y-a)}{x^2+(y-a)^2}\right]_{y=0}$$

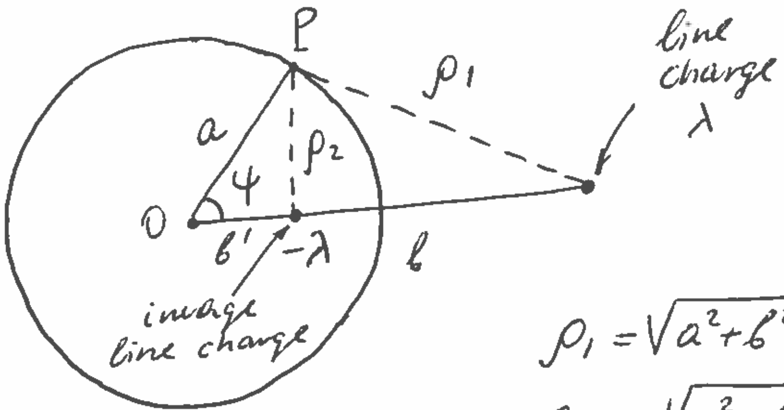
$$= -\frac{\lambda}{4\pi}\frac{4a}{x^2+a^2}$$

$$= -\frac{\lambda a}{\pi(x^2+a^2)} = \underline{\underline{-\frac{\lambda a}{\pi r^2}}},$$

where $r = \sqrt{x^2+a^2}$ is the distance from the point on the plane to the line.

Note that the charge density on the plane is negative (for $\lambda > 0$), since these charges are attracted by the positive charges on the line.

(b) Let us assume that the image charge line with linear charge density $-\lambda$ (to keep the total charge per unit length of the system zero) runs parallel to the axis of the cylinder, at a distance b' from it, as shown on the diagram.



Then the potential at point P on the surface of the cylinder is $\varphi = -\frac{\lambda}{2\pi\varepsilon_0}(\ln\rho_1 - \ln\rho_2)$

$$\rho_1 = \sqrt{a^2+b^2-2ab\cos\psi}$$
$$\rho_2 = \sqrt{a^2+b'^2-2ab'\cos\psi}$$

Hence:

$$\varphi = -\frac{\lambda}{4\pi\varepsilon_0}\ln\frac{a^2+b^2-2ab\cos\psi}{a^2+b'^2-2ab'\cos\psi}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\ln\frac{(a^2+b^2)\left(1-\frac{2ab}{a^2+b^2}\cos\psi\right)}{(a+b'^2)\left(1-\frac{2ab'}{a^2+b'^2}\cos\psi\right)}$$

This potential must be <u>constant</u> on the surface of the cylinder, hence, does not depend on the angle ψ. This is only possible if

$$\frac{2ab}{a^2+b^2} = \frac{2ab'}{a^2+b'^2},$$

so that the brackets containing $\cos\psi$ would cancel.

We have: $(a^2+b'^2)b = b'(a^2+b^2),$

$$b\,b'^2 - (a^2+b^2)\,b' + a^2 b = 0$$

$$b' = \frac{a^2+b^2 \pm \sqrt{(a^2+b^2)^2 - 4a^2b^2}}{2b}$$

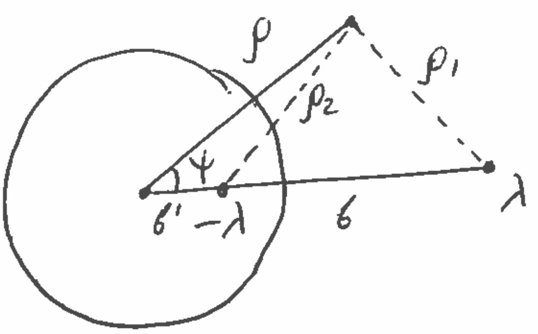
$$= \frac{a^2+b^2 \pm \sqrt{(a^2-b^2)^2}}{2b}$$

$$= \frac{a^2+b^2 \pm (b^2-a^2)}{2b}$$

Recall: $\sqrt{a^2} = |a|$, and $b^2 > a^2$, so $\sqrt{(a^2-b^2)^2} = b^2 - a^2$ here.

Using "+" gives $b' = b$, but using "−" gives $b' = \frac{a^2}{b}$.

This is where the image charge line must be placed.



Now, for any point with cylindrical coordinates (ρ, φ), we have:

$$\Phi(\rho,\varphi) = -\frac{\lambda}{2\pi\varepsilon_0}\left(\ln\rho_1 - \ln\rho_2\right)$$

$$= -\frac{\lambda}{2\pi\varepsilon_0}\left[\ln\sqrt{\rho^2+b^2-2\rho b\cos\varphi} - \ln\sqrt{\rho^2+b'^2-2\rho b'\cos\varphi}\right]$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\ln\frac{\rho^2+b^2-2\rho b\cos\varphi}{\rho^2+\frac{a^4}{b^2}-2\rho\frac{a^2}{b}\cos\varphi}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\ln\frac{(\rho^2+b^2-2\rho b\cos\varphi)\,b^2/a^2}{\frac{b^2}{a^2}\left(\rho^2+\frac{a^4}{b^2}-2\rho\frac{a^2}{b}\cos\varphi\right)}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\left[\ln\frac{\rho^2+b^2-2\rho b\cos\varphi}{(\rho b/a)^2+a^2-2\rho b\cos\varphi} + \ln\frac{b^2}{a^2}\right]$$

This is a constant term, which can be dropped from Φ.

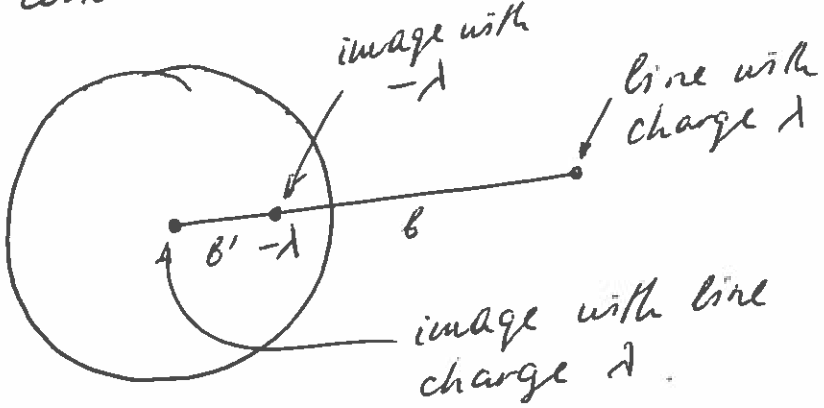
Hence :

$$\varphi(\rho,\psi) = -\frac{\lambda}{4\pi\varepsilon_0}\ln\frac{\rho^2+b^2-2\rho b\cos\psi}{(\rho b/a)^2+a^2-2\rho b\cos\psi},$$

as required.

(c) By Gauss's law, the cylinder whose potential (outside it) is given by the potential of linear charge with density $-\lambda$, carries linear charge $-\lambda$ (i.e., charge per unit length).

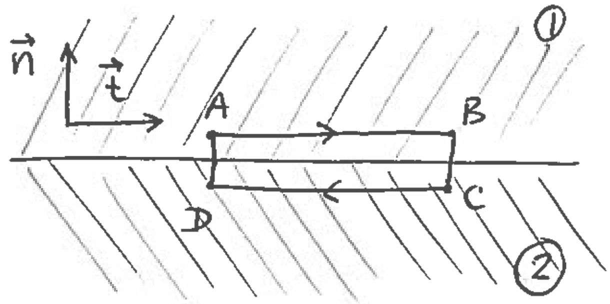
In order to describe the potential of an <u>unchaged</u> cylinder, we must place an additional image charge in the form of a line with density λ <u>on the axis</u> of the cylinder. Adding this charge does not change the condition $\varphi = \text{const}$ on the surface of the cylinder.



The potential of this system is given by

$$\varphi(\rho,\psi) = -\frac{\lambda}{4\pi\varepsilon_0}\ln\frac{\rho^2+b^2-2\rho b\cos\psi}{(\rho b/a)^2+a^2-2\rho b\cos\psi} \underbrace{-\frac{\lambda}{2\pi\varepsilon_0}\ln\rho}_{\text{potential of the extra image line charge.}}$$

4 (a)



We know that $\oint_L \vec{E}\cdot d\vec{r} = 0$ for any closed contour L. Let us choose the contour as a rectangle whose long sides are parallel to the boundary surface, and short sides perpendicular to it.

This rectangle ABCD is shown in the diagram.
Regarding the sides BC and AD as infinitesimal, the only nonzero contributions to the integral come from AB and DC. If we assume that the rectangle is sufficiently small, so that the electric field does not change along AB or DC, we have:

$$\oint_L \vec{E}\cdot d\vec{r} = \int_{AB} \vec{E}\cdot d\vec{r} + \int_{CD} \vec{E}\cdot d\vec{r}$$

$$= \int_{AB} \vec{E}\cdot\vec{t}\,dx + \int_{CD} \vec{E}\cdot(-\vec{t}\,dx)$$

$\vec{t}$ is unit vector tangential to the surface

[where x is the coordinate along AB or CD]

$$= \int_{AB} E_{1t}\,dx - \int_{CD} E_{2t}\,dx$$

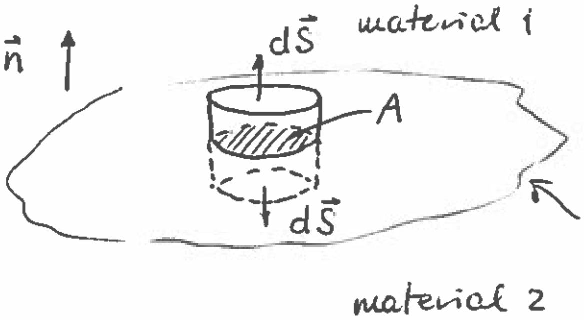
$$= E_{1t}\int_{AB} dx - E_{2t}\int_{CD} dx$$

$$= (E_{1t} - E_{2t})\,l\,,$$ where $l = \int_{AB} dx = \int_{CD} dx$ is the length of AB or CD.

Since this quantity is zero,

we must have $\underline{\underline{E_{1t} = E_{2t}}}$.

To derive the second boundary condition we apply Gauss's law $\oint_S \vec{D}\cdot d\vec{S} = Q_f$ to a cylindrical surface whose bases are parallel to the boundary surface. We also make the length of the cylinder very small, so that the bases are very close to the boundary surface. In this case the flux of $\vec{D}$ is given by the contribution of two bases:



$$\oint_S \vec{D}\cdot d\vec{S} = \int_{\substack{\text{top}\\ \text{base}}} \vec{D}\cdot d\vec{S} + \int_{\substack{\text{bottom}\\ \text{base}}} \vec{D}\cdot d\vec{S}$$

$$= \int_{\substack{\text{top}\\ \text{base}}} \vec{D}\cdot\vec{n}\,dS + \int_{\substack{\text{bottom}\\ \text{base}}} \vec{D}\cdot(-\vec{n}\,dS)$$

$$= \int_{\substack{\text{top}\\ \text{base}}} D_n\,dS - \int_{\substack{\text{bottom}\\ \text{base}}} D_n\,dS$$

free charge inside the cylinder

$$= D_{1n}A - D_{2n}A = \sigma A, \qquad (*)$$

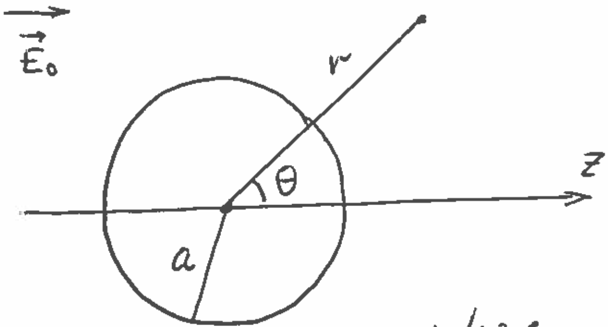
where A is the area of each of the bases, as well as the area of the boundary surface inside the cylinder. In taking the 2nd last step we also assumed that the cylinder is sufficiently small so that D_n does not change over its top or bottom surface.

From $(*)$ we have: $D_{1n} - D_{2n} = \sigma$, or

$$\underline{\underline{D_{1n} = D_{2n} + \sigma}},$$

where D_{1n} and D_{2n} are the normal components of the displacement vector in medium 1 and medium 2, respectively.

(b) The electrostatic potential of the system outside the conducting sphere satisfies Laplace's equation $\nabla^2 \varphi = 0$. Choosing the z axis along the direction of the external field, we can write the general axially symmetric solution in spherical coordinates (r, θ) as



$$\varphi(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta),$$

where A_n and B_n are constants, and $P_n(\cos\theta)$ are Legendre polynomials.

This potential must satisfy the boundary conditions:

(1) For $r \gg a$ $\quad \varphi \simeq -E_0 z = -E_0 r\cos\theta$

(2) On the surface of the conducting sphere

$$\varphi(a,\theta) = \varphi_a = \text{const}.$$

Condition (1) shows that $A_n = 0$ for $n \geq 2$, so that

$$\varphi(r,\theta) = A_0 + A_1 r\cos\theta + \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta),$$

where we take into account $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$.

Also, to satisfy (1) we must have $A_1 = -E_0$, so that

$$\varphi(r,\theta) = A_0 - E_0 r\cos\theta + \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

The second boundary condition then gives:

$$A_0 - E_0 a\cos\theta + \sum_{n=0}^{\infty} \frac{B_n}{a^{n+1}} P_n(\cos\theta) = \varphi_a .$$

For this to hold for all θ, the coefficients which multiply $P_n(\cos\theta)$ on both sides should be equal. Hence:

$n=0$: $A_0 + \frac{B_0}{a} = \varphi_a \Rightarrow \underline{A_0 = \varphi_a - \frac{B_0}{a}}$.

$n=1$: $-E_0 a + \frac{B_1}{a^2} = 0 \Rightarrow \underline{B_1 = E_0 a^3}$

$n \geq 2$: $\frac{B_n}{a^{n+1}} = 0 \Rightarrow \underline{B_n = 0}$.

Hence, the potential outside the sphere is

$$\varphi(r,\theta) = \varphi_a - \frac{B_0}{a} - E_0 r\cos\theta + \frac{B_0}{r} + \frac{E_0 a^3}{r^2}\cos\theta .$$

We are given that the total charge on the sphere is Q. The surface charge density can be found as

$$\sigma = -\varepsilon_0 \frac{\partial \varphi}{\partial n} = -\varepsilon_0 \frac{\partial \varphi}{\partial r}\Big|_{r=a}$$

$$= -\varepsilon_0 \left[-E_0\cos\theta - \frac{B_0}{a^2} - \frac{2E_0 a^3}{a^3}\cos\theta\right]$$

$$= -\varepsilon_0 \left[-3E_0\cos\theta - \frac{B_0}{a^2}\right]$$

$$= 3\varepsilon_0 E_0\cos\theta + \frac{B_0\varepsilon_0}{a^2} \qquad (1)$$

The total charge is found by integrating over the surface of the sphere:

$$Q = \int \sigma\, dS = \int_0^{2\pi}\int_0^{\pi} \left(3\varepsilon_0 E_0 \cos\theta + \frac{B_0\varepsilon_0}{a^2}\right) \underbrace{a^2 \sin\theta\, d\theta\, d\varphi}_{\substack{dS \text{ in} \\ \text{spherical} \\ \text{coordinates}}}$$

$$Q = a^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \left(3\varepsilon_0 E_0 \cos\theta + \frac{B_0 \varepsilon_0}{a^2}\right) \sin\theta \, d\theta$$

$$= 2\pi a^2 \left[3\varepsilon_0 E_0 \int_0^{\pi} \cos\theta \sin\theta \, d\theta + \frac{B_0 \varepsilon_0}{a^2} \int_0^{\pi} \sin\theta \, d\theta \right]$$

$$= 2\pi a^2 \left[3\varepsilon_0 E_0 \int_0^{\pi} \cos\theta \, (-1) \, d(\cos\theta) + \frac{B_0 \varepsilon_0}{a^2} \left[-\cos\theta\right]_0^{\pi} \right]$$

$$= 2\pi a^2 \left[3\varepsilon_0 E_0 \left[-\frac{\cos^2\theta}{2}\right]_0^{\pi} + \frac{B_0 \varepsilon_0}{a^2} \left[-(-1) - (-1)\right] \right]$$

$$= 2\pi a^2 \left[3\varepsilon_0 E_0 \left(-\tfrac{1}{2} + \tfrac{1}{2}\right) + \frac{2 B_0 \varepsilon_0}{a^2} \right]$$

$$= 4\pi \varepsilon_0 B_0 \quad \Rightarrow \quad B_0 = \frac{Q}{4\pi\varepsilon_0} .$$

Hence, the potential is

$$\varphi(r,\theta) = \varphi_a - \frac{Q}{4\pi\varepsilon_0 a} - E_0 r \cos\theta + \frac{Q}{4\pi\varepsilon_0 r} + \frac{E_0 a^3}{r^2} \cos\theta$$

We can use the freedom of adding a constant to the potential (or, equivalently, choosing φ_a) and write

$$\varphi(r,\theta) = -E_0 r \cos\theta + \frac{Q}{4\pi\varepsilon_0 r} + \frac{E a^3}{r^2} \cos\theta . \qquad (2)$$

The corresponding surface charge density, from equation (1) is

$$\sigma = 3\varepsilon_0 E_0 \cos\theta + \frac{Q}{4\pi a^2} .$$

Here the first term is due charges induced on the surface of the sphere by the external field, and the second term is due to the total charge Q of the sphere.

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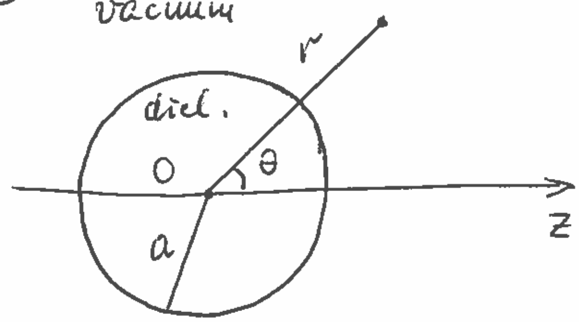


Let the potential in vacuum $(r>a)$ be

$$\varphi_v(r,\theta) = \frac{A_v}{r^2}\cos\theta + \frac{B_v}{r} + C_v + D_v r\cos\theta$$

and in the dielectric $(r<a)$

$$\varphi_d(r,\theta) = \frac{A_d}{r^2}\cos\theta + \frac{B_d}{r} + C_d + D_d r\cos\theta .$$

Let us use the various boundary conditions to determine the values of the constants.

1) There is no singularity at the centre of the sphere. Hence, $A_d = 0$, $B_d = 0$

$$\Rightarrow \quad \varphi_d(r,\theta) = C_d + D_d r\cos\theta . \qquad (1)$$

2) At large distances, $r \gg a$, the field is uniform, in the z direction, so we must have

$$\varphi \simeq -E_0 z = -E_0 r\cos\theta .$$

Comparing with $\varphi_v(r,\theta)$ above we see that $D_v = -E_0$,

so

$$\varphi_v(r,\theta) = C_v - E_0 r\cos\theta + \frac{A_v}{r^2}\cos\theta + \frac{B_v}{r} . \qquad (2)$$

3) On the surface of the sphere the potentials are equal:

$$\varphi_d(a,\theta) = \varphi_v(r,\theta),$$

or

$$C_d + D_d a\cos\theta = C_v - E_0 a\cos\theta + \frac{A_v}{a^2}\cos\theta + \frac{B_v}{a}$$

Hence

$$C_d = C_v + \frac{B_v}{a} \qquad (3)$$

$$D_d a = -E_0 a + \frac{A_v}{a^2} \qquad (4)$$

4) There are no free charges on the surface of the sphere, so that $D_{dn} = D_{vn}$ (normal components of the electric displacement)

Using $\vec{D} = \varepsilon_0 \chi \vec{E}$, we have:

$$\varepsilon_0 \chi E_{dn} = \varepsilon_0 E_{vn}$$

or $$\chi \left(-\frac{\partial \varphi_d}{\partial n} \right) = -\frac{\partial \varphi_v}{\partial n},$$ which gives

$$\chi \left. \frac{\partial \varphi_d}{\partial r} \right|_{r=a} = \left. \frac{\partial \varphi_v}{\partial r} \right|_{r=a},$$ since the normal is in the direction of r.

Hence, from (1) and (2),

$$\chi D_d \cos\theta = -E_0 \cos\theta - \frac{2A_v}{a^3}\cos\theta - \frac{B_v}{a^2}.$$

This gives $$0 = -\frac{B_v}{a^2} \qquad (5)$$

and $$\chi D_d = -E_0 - \frac{2A_v}{a^3} \qquad (6).$$

From (5), $\underline{\underline{B_v = 0}}$, and then, from (3),

$$\underline{\underline{C_d = C_v \equiv C}}.$$

Solving (4) and (6) simultaneously:

$$\left.\begin{aligned} D_d - \frac{A_v}{a^3} &= -E_0 \\ \chi D_d + \frac{2A_v}{a^2} &= -E_0 \end{aligned}\right\} \begin{matrix} \times 2 \\ + \end{matrix} \;\Bigg|\; \begin{matrix} \times(-\chi) \\ + \end{matrix}$$

$$(2+\chi) D_d = -3E_0 \Rightarrow \underline{\underline{D_d = -\frac{3}{2+\chi} E_0}}.$$

$$\frac{\chi A_v}{a^3} + \frac{2A_v}{a^3} = \chi E_0 - E_0$$

$$\frac{(2+\chi) A_v}{a^3} = (\chi - 1) E_0 \Rightarrow \underline{\underline{A_v = \frac{\chi - 1}{2+\chi} E_0 a^3}}$$

Hence, the potential is:

$$\varphi(r,\theta) = C - \frac{3}{2+x} E_0 r \cos\theta \qquad (r<a)$$

$$\varphi(r,\theta) = C - E_0 r\cos\theta + \frac{x-1}{2+x} E_0 a^3 \frac{\cos\theta}{r^2} \qquad (r>a)$$

To find the volume and surface polarisation charge densities, we use

$$\vec{\nabla}\cdot\vec{P} = -\rho_p \quad \text{and} \quad P_n = \sigma_p ,$$

where $\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \varepsilon_0 x \vec{E} - \varepsilon_0 \vec{E} = \varepsilon_0 (x-1)\vec{E}$

$$\Rightarrow \quad \vec{P} = -\varepsilon_0 (x-1)\vec{\nabla}\varphi .$$

For $r<a$, $\varphi(r,\theta) = C - \frac{3}{2+x} E_0 r\cos\theta$

$$= C - \frac{3}{2+x} E_0 z .$$

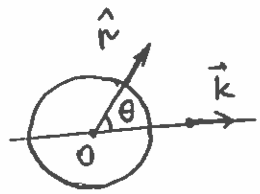
Hence $\vec{\nabla}\varphi = -\frac{3}{2+x} E_0 \vec{k}$,

and $\vec{P} = \frac{3(x-1)}{2+x} \varepsilon_0 E_0 \vec{k}$.

This is a constant vector so that $\vec{\nabla}\cdot\vec{P} = 0$ and the volume polarisation charge density $\underline{\underline{\rho_p = 0}}$.

The surface polarisation charge density

$$\sigma_p = P_n = \vec{P}\cdot\hat{r} = \frac{3(x-1)}{2+x}\varepsilon_0 E_0 \underbrace{\vec{k}\cdot\hat{r}}_{=\cos\theta}$$



$$\Rightarrow \quad \underline{\underline{\sigma_p = \frac{3(x-1)}{2+x}\varepsilon_0 E_0 \cos\theta}} .$$

Naturally, we observe $\sigma_p > 0$ for $0 \le \theta < \frac{\pi}{2}$

and $\sigma_p < 0$ for $\frac{\pi}{2} < \theta \le \pi$.

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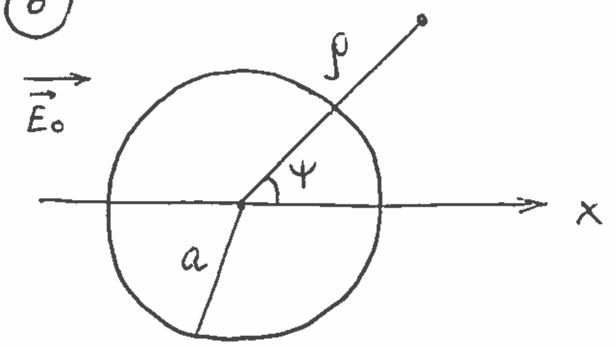

The potential of the cylinder in the external field should satisfy the conditions:

(1) For $\rho \gg a$

$$\varphi \simeq -E_0 x = -E_0 \rho \cos\psi ,$$

where the x axis is along the external field $\vec{E}_0$

(2) $\varphi = \text{const}$ on the surface of the cylinder.

(3) Total charge per unit length of the cylinder is zero

From (1) we see that in the general solution

$$\varphi(\rho,\psi) = C \ln\rho + \sum_{n=-\infty}^{\infty} (A_n \cos n\psi + B_n \sin n\psi)\rho^n$$

all coefficients A_n and B_n with $n \geq 2$ vanish must be zero and that $B_1 = 0$, while

$$A_1 = -E_0 .$$

Hence:

$$\varphi(\rho,\psi) = C \ln\rho - E_0 \rho \cos\psi + \sum_{n=-\infty}^{0} (A_n \cos n\psi + B_n \sin n\psi) \times \rho^n$$

From (2):

$$\varphi(a,\psi) = C \ln a - E_0 a \cos\psi + \sum_{n=-\infty}^{0} (A_n \cos n\psi + B_n \sin n\psi) a^n = \varphi_a$$

The coefficients which multiply $\cos n\psi$ or $\sin n\psi$ on both sides of this equation must be equal. Hence, for $n = 0$: $\quad C \ln a + A_0 = \varphi_a \quad (1)$

and B_0 can be set to 0 since $\sin 0\psi = 0$.

$n = 1, -1 \qquad -E_0 a + A_{-1} a^{-1} = 0 \qquad (2)$

(note: $\cos(-\varphi) = \cos\varphi$)

and $\qquad \underline{B_{-1} = 0}$

$n \leq -2 \qquad \underline{A_n = B_n = 0}$.

From (2)

$$\underline{A_{-1} = E_0 a^2}.$$

From (1)

$$\underline{A_0 = \varphi_a - C \ln a}.$$

Hence, the potential is given by

$$\varphi(\rho, \varphi) = C \ln\rho + \underbrace{\varphi_a - C\ln a}_{\text{This is a constant term which can be chosen arbitrarily, e.g. } \varphi_a = C\ln a \text{ can be imposed, or we can set } \varphi_a = 0.} - E_0 \rho\cos\varphi + E_0 a^2 \frac{\cos\varphi}{\rho}$$

Let us find the surface charge density on the cylinder.

$$\sigma = -\varepsilon_0 \frac{\partial\varphi}{\partial n} = -\varepsilon_0 \left.\frac{\partial\varphi}{\partial\rho}\right|_{\rho=a}$$

$$= -\varepsilon_0 \left[C\frac{1}{a} - E_0\cos\varphi - E_0 a^2 \frac{\cos\varphi}{a^2} \right]$$

$$= -\varepsilon_0 C \frac{1}{a} + 2\varepsilon_0 E_0 \cos\varphi.$$

The charge per unit length of the cylinder is found as

$$\int \sigma\, dS = \int_0^{2\pi} \sigma \cdot 1\, a\, d\varphi = \int_0^{2\pi} a\left(-\varepsilon_0 \frac{C}{a} + 2\varepsilon_0 E_0 \cos\varphi\right) d\varphi$$

$$= -\varepsilon_0 C \int_0^{2\pi} d\varphi + 2\varepsilon_0 E_0 a \int_0^{2\pi} \cos\varphi \, d\varphi$$

$$= -2\pi\varepsilon_0 C + 2\varepsilon_0 E_0 a \left[\sin\varphi\right]_0^{2\pi} = -2\pi\varepsilon_0 C .$$

Hence, since the cylinder is uncharged, we must set $C = 0$.

The potential outside the cylinder is then given by

$$\underline{\underline{\varphi(\rho,\varphi) = -E_0 \rho \cos\varphi + E_0 a^2 \frac{\cos\varphi}{\rho}}},$$

and the surface charge density is

$$\underline{\underline{\sigma = 2\varepsilon_0 E_0 \cos\varphi}}.$$

It is maximum for $\varphi = 0$ (the direction of the external field), so that

$$\underline{\underline{\sigma_{max} = 2\varepsilon_0 E_0}}.$$