Electrostatic energy. Steady currents.

Homework

1. (a) Show that the electrostatic energy stored in a condenser of capacitance C is

$$U = \frac{1}{2}CV^2, \quad \text{or equivalently,} \quad U = \frac{Q^2}{2C},$$

where Q is the charge on the plates and V is the potential difference between them.

(b) Derive this result for a parallel-plate capacitor filled by dielectric with permittivity ε, using the energy density of the electrostatic field, $\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}$, and neglecting edge effects.

2. Consider a ball of radius R charged uniformly with the volume charge density ρ, so that its total charge is $Q = 4\pi R^3\rho/3$.

(a) Use Gauss's law to find ϕ, $\mathbf{E}$ and $\mathbf{D}$ everywhere. Express your answers both in terms of ρ and Q.

(b) Determine the electrostatic energy of the system in two ways, i.e., using

$$U = \frac{1}{2}\int_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})dV + \frac{1}{2}\int_S \sigma(\boldsymbol{r})\phi(\boldsymbol{r})dS, \tag{1}$$

and using

$$U = \frac{1}{2}\int_V \mathbf{E}\cdot\mathbf{D}dV. \tag{2}$$

(c) Express the energy U from part (b) in terms of Q and R and compare it with the electrostatic energy of a metallic sphere of radius R carrying charge Q.

[Note that in part (b), $\sigma = 0$ in (1), and that the integral in (2) is over the whole space.]

3. The plates of a parallel plate condenser have length a, width b and are separated by a distance d. A block of solid dielectric of permittivity ε fills the region between them. The block is withdrawn along a so that only length x of it remains between the plates.

(a) Assuming that the potential difference between the plates is kept at the value V, show that the force on the block is

$$F_x = \frac{V^2 b}{2d}(\varepsilon - \varepsilon_0),$$

and that this force acts so as to pull it into the condenser.

(b) Assuming that the condenser is isolated and carries total charge Q, show that the force on the block is

$$F_x = \frac{Q^2 d(\varepsilon - \varepsilon_0)}{2b[\varepsilon x + \varepsilon_0(a - x)]^2}.$$

(c) Verify that the force is the same in the two cases if the potential difference between the plates is the same.

4. Electric current flows through a medium of uniform conductivity σ between two concentric spherical electrodes of radii a and b, where $b > a$. The potential difference between the electrodes is V. Show that the current density at the outer electrode is

$$\boldsymbol{j} = \sigma \frac{a}{b(b-a)} V \hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{r}}$ is the radially outward unit vector.

5. The space between two coaxial cylinders of length l and radii a and b, where $b > a$, is filled with a medium of conductivity σ.

   (a) Show that the resistance between the two cylinders is

   $$R = \frac{1}{2\pi\sigma l} \ln \frac{b}{a}.$$

   (b) Show that for $b - a \equiv d \ll a$ the above formula gives $R = d/\sigma A$ with $A = 2\pi a l$, as one would expected for a uniform resistor of length d and cross sectional area A.

   [Hint: use the Maclaurin expansion to first order, $\ln(1 + x) \simeq x$.]

6. (a) Derive the continuity equation for the conservation of electric charge.

   (b) Show that, in a homogeneous isotropic conductor satisfying Ohm's law, the flow of steady current can be analysed using Laplace's equation.

   (c) Write down the solution to Laplace's equation in cylindrical polar coordinates (ρ, ψ, z) that is independent of ρ and z.

   (d) A semicylinder with inner radius a and outer radius b is made out of material of uniform conductivity σ and occupies the region $a \leq \rho \leq b$, $0 \leq \psi \leq \pi$, $0 \leq z \leq h$, in cylindrical polar coordinates. A steady current enters the $\psi = 0$ face at potential ϕ_1 and leaves the $\psi = \pi$ face at potential ϕ_2. You may assume that the potential only depends on ψ.

      i. Find the total current flowing and the total resistance R.

      ii. Verify your results by showing that $R \simeq l/\sigma A$, where l is the length of the resistor and A is its cross section, in the limit as b and a tend to infinity keeping $b - a$ fixed.

   [Exam 2006 Question 4]

Further Examples

1. S_1 is a fixed hollow conducting cylinder of radius b, and S_2 is a smaller solid conducting cylinder of radius a, coaxial with S_1. S_2 is free to slide along its axis and a constant potential difference V is maintained between the two cylinders.

   Show that when a length l of S_2 is inside S_1, the electrostatic energy is

   $$U = \frac{\pi\varepsilon_0 l V^2}{\ln(b/a)},$$

   and hence deduce that the movable cylinder experiences a force

   $$F = \frac{\pi\varepsilon_0 V^2}{\ln(b/a)},$$

   drawing it inside the fixed cylinder.

2. A homogeneous isotropic conducting material of conductivity σ occupies the volume

   $$a \le r \le b, \qquad \frac{\pi}{3} \le \theta \le \frac{\pi}{2}, \qquad 0 \le \psi \le 2\pi,$$

   where (r, θ, ψ) are spherical polar coordinates. If the plane boundary $\theta = \pi/2$ and the conical boundary $\theta = \pi/3$ are used as electrodes (assumed to be equipotentials), show that the resistance of this conducting body is

   $$R = \frac{\ln 3}{4\pi\sigma(b-a)}.$$

3. An infinite uniform conducting sheet of conductivity σ lies in the plane $z = 0$ of a Cartesian coordinate system with origin at O and carries a uniform current of density $\boldsymbol{j}_0$ in the direction of the x axis.

   (a) Show that the potential drop from $x = -a$, $y = 0$ to $x = a$, $y = 0$ is $2j_0 a/\sigma$.

   (b) If a circular hole of radius a centred at O is cut from the sheet, show that the potential drop from $x = -a$, $y = 0$ to $x = a$, $y = 0$ now becomes $4j_0 a/\sigma$, and that the new current density becomes

   $$\boldsymbol{j} = j_0 \cos\psi \left(1 - \frac{a^2}{\rho^2}\right) \hat{\boldsymbol{\rho}} - j_0 \sin\psi \left(1 + \frac{a^2}{\rho^2}\right) \hat{\boldsymbol{\psi}},$$

   where (ρ, ψ, z) are cylindrical polar coordinates with O as origin.