AMA3001 Electromagnetic Theory

Problem sheet 4

SOLUTIONS

1 (a) For a system of conductors with charges Q_i and potentials φ_i, the electrostatic energy is

$$U = \frac{1}{2} \sum_i Q_i \varphi_i .$$

A capacitor consists of two conductors (plates) with charges Q and $-Q$. Hence,

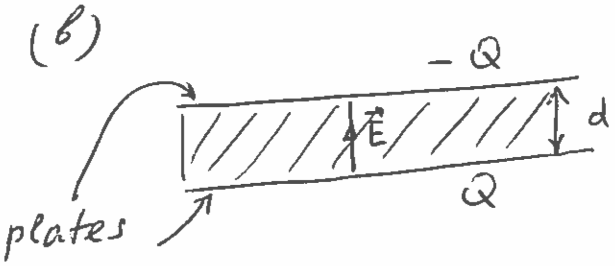
$$U = \frac{1}{2}(Q\varphi_1 - Q\varphi_2) = \frac{1}{2} Q(\varphi_1 - \varphi_2) = \frac{1}{2} QV,$$

where $V = \varphi_1 - \varphi_2$ is the potential difference.
Using the definition of capacitance, $C = Q/V$, we have $Q = CV$, so that

$$U = \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{Q^2}{2C},$$

where we used $V = Q/C$ in the last step.

(b)



Inside the capacitor the field is uniform and perpendicular to the plates, and $E = (\varphi_1 - \varphi_2)/d = V/d$.

(The latter follows from $\varphi_2 - \varphi_1 = \int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla}\varphi \cdot d\vec{r} = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$)

$$\vec{D} = \varepsilon \vec{E}, \quad \text{so} \quad \frac{1}{2} \vec{E} \cdot \vec{D} = \frac{1}{2} \varepsilon E^2 .$$

The total electrostatic energy is obtained by integrating over the volume of the dielectric inside the capacitor:

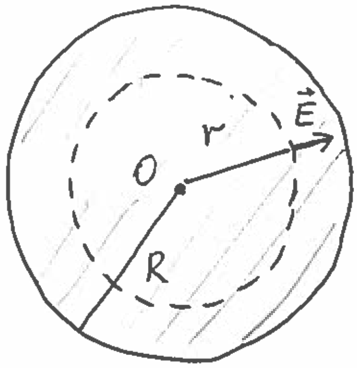
$$U = \frac{1}{2} \int_V \vec{E} \cdot \vec{D}\, dV = \frac{1}{2} \varepsilon E^2 A d,$$ where A is the area

of each of the plates and Ad is the volume inside the capacitor. Using $E = V/d$, we have:

$$U = \frac{1}{2}\varepsilon\frac{V^2}{d^2}Ad = \frac{1}{2}\frac{\varepsilon A}{d}V^2 = \underline{\underline{\frac{1}{2}CV^2}},$$

where $C = \frac{\varepsilon A}{d} = \frac{\varepsilon_0 \varkappa A}{d}$ is the capacitance of the parallel-plate capacitor with permittivity ε and dielectric constant $\varkappa$.

② (a)



Due to the symmetry of the system, the electric field is along the radius and its magnitude depends on r only:

$$\vec{E}(\vec{r}) = E(r)\,\hat{r}.$$

Applying Gauss's law

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{Q_r}{\varepsilon_0}$$

to the sphere of radius r, $r<R$, where $Q_r = \frac{4}{3}\pi r^3\rho$ is the charge inside such sphere, we have:

$$E(r)4\pi r^2 = \frac{4}{3\varepsilon_0}\pi r^3\rho$$

$$\Rightarrow \quad \underline{\underline{E(r) = \frac{\rho r}{3\varepsilon_0}}} \quad \text{and} \quad \underline{\underline{\vec{E} = \frac{\rho}{3\varepsilon_0}\vec{r}}}. \qquad (1)$$

The corresponding displacement is

$$\underline{\underline{\vec{D} = \varepsilon_0\vec{E} = \frac{\rho}{3}\vec{r}}}. \qquad (1')$$

Applying Gauss's law to the sphere of radius $r>R$, we have: $E(r)\,4\pi r^2 = \frac{Q}{\varepsilon_0}$, where $Q = \frac{4}{3}\pi R^3 \rho$ is the total charge of the ball.

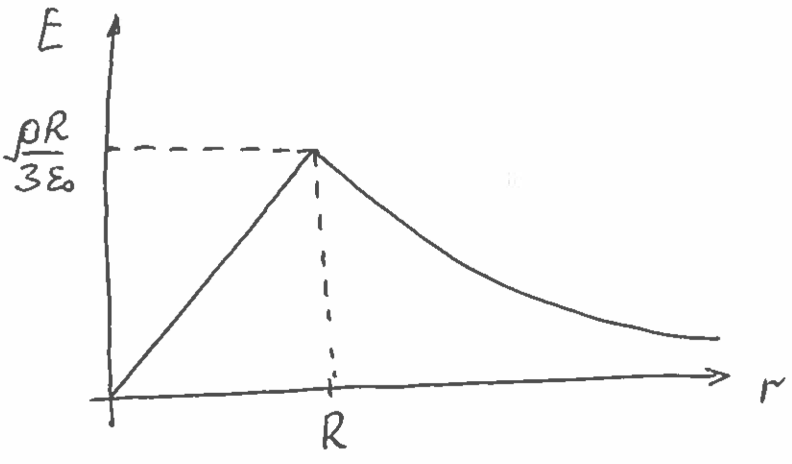
Therefore,

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow \quad \underline{\underline{\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}}}, \qquad (2)$$

and the displacement is

$$\vec{D} = \varepsilon_0 \vec{E} = \underline{\underline{\frac{Q}{4\pi}\frac{\hat{r}}{r^2}}}. \qquad (2')$$

[Graphically (though this is not required by the question)



Note that if a straight tunnel was dug through the Earth's centre, the gravitational force acting on an object would have looked the same, with the max. value of mg at the Earth's surface.]

Written in terms of Q, i.e. using $\rho = \frac{3Q}{4\pi R^3}$,

(1) and (1') read:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^3}\vec{r}, \qquad \vec{D} = \frac{Q}{4\pi R^3}\vec{r} \qquad (r<R) \; (1'')$$

while (2) and (2'), written in terms of ρ, give:

$$\vec{E} = \frac{R^3\rho}{3\varepsilon_0}\frac{\hat{r}}{r^2}, \qquad \vec{D} = \frac{R^3\rho}{3}\frac{\hat{r}}{r^2}. \qquad (2'')$$

Using the relation (in spherical coordinates),

$$\vec{E} = -\vec{\nabla}\varphi = -\frac{\partial\varphi}{\partial r}\hat{r}, \quad \text{so that}$$

$$E(r) = -\frac{d\varphi}{dr} \qquad (\varphi \text{ depends on } r \text{ only}),$$

we find

$$\varphi = -\int E(r)\,dr .$$

Outside the sphere,

$$\varphi(r) = -\int \frac{Q}{4\pi\varepsilon_0 r^2}\,dr = \frac{Q}{4\pi\varepsilon r} + \text{const}$$

Setting const $= 0$, so that $\varphi \to 0$ at infinity,

we have $$\varphi(r) = \frac{Q}{4\pi\varepsilon_0 r} \qquad (r > R). \qquad (3)$$

Inside the ball:

$$\varphi(r) = -\int \frac{\rho r}{3\varepsilon_0}\,dr = -\frac{\rho r^2}{6\varepsilon_0} + C.$$

The constant C must be chosen in such a way that for $r = R$ the potential had the same value as given by (3):

$$-\frac{\rho R^2}{6\varepsilon_0} + C = \frac{Q}{4\pi\varepsilon_0 R},$$

$$-\frac{QR^2}{\frac{4}{3}\pi R^3\, 6\varepsilon_0} + C = \frac{Q}{4\pi\varepsilon_0 R}$$

$$-\frac{Q}{8\pi\varepsilon_0 R} + C = \frac{Q}{4\pi\varepsilon_0 R}$$

$$\Rightarrow \qquad C = \frac{3Q}{8\pi\varepsilon_0 R}$$

and $$\varphi = \frac{3Q}{8\pi\varepsilon_0 R} - \frac{\rho r^2}{6\varepsilon_0}$$

$$= \frac{3Q}{8\pi\varepsilon_0 R} - \frac{Qr^2}{8\pi\varepsilon_0 R^3}$$

$$= \frac{3Q}{8\pi\varepsilon_0 R}\left(1 - \frac{r^2}{3R^2}\right) \qquad (r < R) \qquad (4)$$

Or, in terms of ρ:

$$\varphi = \frac{\rho R^2}{2\varepsilon_0}\left(1 - \frac{r^2}{3R^2}\right). \qquad (4')$$

(b) In the absence of surface charges, the electrostatic energy is

$$U = \frac{1}{2}\int_V \rho\varphi\, dV, \quad \text{and using} \quad (4')$$

$$= \frac{1}{2}\int_0^R \rho \;\frac{\rho R^2}{2\varepsilon_0}\left(1-\frac{r^2}{3R^2}\right)\underbrace{4\pi r^2 dr}$$

dV - volume of spherical shell of radius r and thickness dr

[$\rho = \text{const}$, the integral is over the volume of the ball.]

$$= 4\pi\frac{\rho^2 R^2}{4\varepsilon_0}\int_0^R\left(r^2 - \frac{r^4}{3R^2}\right)dr$$

$$= \frac{\pi\rho^2 R^2}{\varepsilon_0}\left[\frac{r^3}{3} - \frac{r^5}{15R^2}\right]_0^R$$

$$= \frac{\pi\rho^2 R^2}{\varepsilon_0}\left(\frac{R^3}{3} - \frac{R^3}{15}\right)$$

$$\Rightarrow \quad \underline{\underline{U = \frac{4\pi\rho^2 R^5}{15\varepsilon_0}}}.$$

Using the second formula for U, together with equations (1), (1') and (2'''), we have:

$$U = \frac{1}{2}\int_V \vec{E}\cdot\vec{D}\, dV$$

$$= \frac{1}{2}\int_0^R \frac{\rho r}{3\varepsilon_0}\cdot\frac{\rho r}{3}\, 4\pi r^2 dr + \frac{1}{2}\int_R^\infty \frac{R^3\rho}{3\varepsilon_0 r^2}\,\frac{R^3\rho}{3r^2}\, 4\pi r^2 dr$$

$$= \frac{2\pi\rho^2}{9\varepsilon_0}\int_0^R r^4 dr + \frac{2\pi R^6\rho^2}{9\varepsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$= \frac{2\pi\rho^2}{9\varepsilon_0}\left[\frac{r^5}{5}\right]_0^R + \frac{2\pi R^6\rho^2}{9\varepsilon_0}\left[-\frac{1}{r}\right]_R^\infty$$

$$= \frac{2\pi \rho^2 R^5}{45\,\varepsilon_0} + \frac{2\pi\rho^2 R^5}{9\,\varepsilon_0} = \frac{2\pi\,(1+5)\,\rho^2 R^5}{45\,\varepsilon_0}$$

$$\Rightarrow \quad U = \frac{4\pi\,\rho^2 R^5}{15\,\varepsilon_0},$$

which is the same as obtained using the 1st formula.

(c) In terms of Q, $\rho = \frac{3Q}{4\pi R^3}$, so that

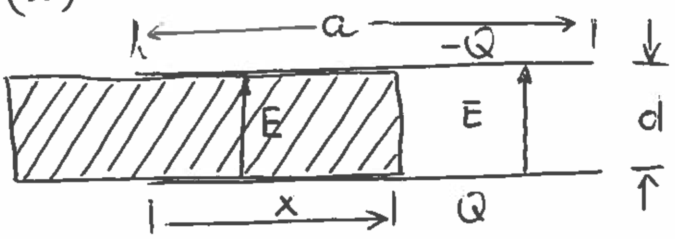
$$U = \frac{4\pi R^5}{15\,\varepsilon_0}\cdot\frac{9Q^2}{(4\pi)^2 R^6} = \frac{3}{5}\cdot\frac{Q^2}{4\pi\varepsilon_0 R}.$$

For a metallic sphere of radius R carrying charge Q,

$$U = \frac{1}{2} Q\varphi = \frac{1}{2} Q \underbrace{\frac{Q}{4\pi\varepsilon_0 R}}_{\substack{\text{Potential} \\ \varphi = \frac{Q}{4\pi\varepsilon_0 r} \\ \text{taken at } r=R}} = \frac{1}{2}\,\frac{Q^2}{4\pi\varepsilon_0 R}.$$

Hence, we see that the energy of the uniformly charged ball is higher than that of the sphere of the same radius, carrying the same charge on the surface. One can say that by moving to the surface the charges are getting as far from each other as possible, thereby lowering the total electrostatic energy. One can also see that for a conducting sphere only the volume outside $(r>R)$ contributes to the integral $\frac{1}{2}\int \vec{E}\cdot\vec{D}\,dV$.

③ (a)



The electric field inside the capacitor is perpendicular to the plates and is related to the potential difference V by

$$E = V/d .$$

Outside the dielectric:

$$D = \varepsilon_0 E$$

Inside the dielectric:

$$D_d = \varepsilon E .$$

Neglecting the edge effects, the electrostatic energy is found as the integral over the volume <u>inside</u> the capacitor:

$$U = \frac{1}{2}\int_V \bar{E}\cdot\bar{D}\, dV = \frac{1}{2} E D \underbrace{(a-x) b d}_{\substack{\text{volume of} \\ \text{the empty} \\ \text{part of the} \\ \text{capacitor}}} + \frac{1}{2} E D_d \underbrace{x b d}_{\substack{\text{volume of} \\ \text{the part} \\ \text{filled with} \\ \text{the dielectric}}}$$

$$\Rightarrow \quad U = \frac{1}{2} \frac{V}{d} \varepsilon_0 \frac{V}{d}(a-x) b d + \frac{1}{2}\frac{V}{d}\varepsilon\frac{V}{d} x b d$$

$$= \frac{1}{2}\frac{[\varepsilon_0(a-x) + \varepsilon x] b}{d} V^2 \qquad (1)$$

Comparing with the formula for the energy of a capacitor, $U = \frac{1}{2} C V^2$,

we see that

$$C = \frac{[\varepsilon_0(a-x) + \varepsilon x] b}{d} \qquad (2)$$

is the capacitance of the capacitor part-filled with the dielectric.

The force acting on the dielectric is in the x direction (because U depends on x) and is found as

$$F_x = \left.\frac{dU}{dX}\right|_{V=\text{const}}$$

$$= \frac{1}{2}\,\frac{(-\varepsilon_0 + \varepsilon)b}{d}\,V^2$$

$$\Rightarrow \quad \underline{\underline{F_x = \frac{1}{2}\,\frac{(\varepsilon - \varepsilon_0)b}{d}\,V^2}}, \quad \text{as required.} \qquad (3)$$

(b) To find the force in the case when the condenser is isolated and $Q = \text{const}$, we express the energy U from (1) in terms of Q rather than V using $V = \frac{Q}{C}$, with C given by (2). This gives:

$$U = \frac{1}{2}\,\frac{[\varepsilon_0(a-x)+\varepsilon x]b}{d}\,\frac{Q^2}{\left(\frac{[\varepsilon_0(a-x)+\varepsilon x]b}{d}\right)^2}$$

$$\Rightarrow \quad U = \frac{1}{2}\,\frac{d}{[\varepsilon_0(a-x)+\varepsilon x]b}\,Q^2 .$$

The force acting on the dielectric is now found from

$$F_x = -\left.\frac{dU}{dX}\right|_{Q=\text{const}} = \underline{\underline{\frac{1}{2}\,\frac{d\,(\varepsilon-\varepsilon_0)}{[\varepsilon_0(a-x)+\varepsilon x]^2 b}\,Q^2}}, \qquad (4)$$

as required.

(c) Substituting $Q = CV = \frac{[\varepsilon_0(a-x)+\varepsilon x]b}{d}\,V$ into (4) gives $U = \frac{1}{2}\,\frac{(\varepsilon-\varepsilon_0)b}{d}\,V^2$; identical to (3).

(4) The system is spherically symmetric, so the potential satisfies $\nabla^2 \varphi = 0$ for $\varphi = \varphi(r)$, which, written in $\underbrace{\text{polar}}_{\text{spherical}}$ coordinates gives:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = 0 .$$

$$\frac{d}{dr}\left(r^2 \frac{d\varphi}{dr}\right) = 0$$

$$r^2 \frac{d\varphi}{dr} = C$$

$$\frac{d\varphi}{dr} = \frac{C}{r^2}$$

$$\varphi = \int \frac{C}{r^2}\, dr$$

$$\varphi = -\frac{C}{r} + D .$$

Potential difference between the two electrodes (sphere of radius a and sphere of radius b) is

$$\varphi(a) - \varphi(b) = -\frac{C}{a} + D - \left(-\frac{C}{b} + D\right)$$

$$= -C\left(\frac{1}{a} - \frac{1}{b}\right) = V$$

$$\Rightarrow \quad C = -\frac{V}{\frac{1}{a} - \frac{1}{b}} = -\frac{abV}{b-a} .$$

Hence, $\varphi(r) = \dfrac{abV}{b-a}\dfrac{1}{r} + D$.

The electric field is in the radial direction

$$\vec{E} = -\vec{\nabla}\varphi = -\frac{\partial \varphi}{\partial r}\hat{r} = \frac{abV}{b-a}\frac{1}{r^2}\hat{r}$$

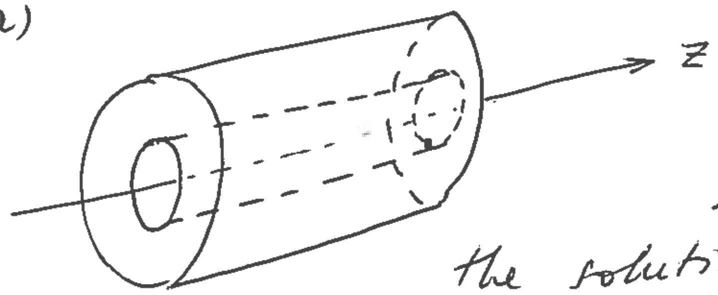
The current density, by Ohm's law, then is

$$\vec{j} = \sigma \vec{E} = \frac{\sigma abV}{b-a}\frac{1}{r^2}\hat{r} .$$

On the outer electrode, $r = b$

$$\bar{j} = \frac{\sigma a b V}{(b-a) b^2} \hat{r} = \underline{\underline{\frac{\sigma a V}{b(b-a)} \hat{r}}}, \text{ as required.}$$

⑤ (a)



This system has cylindrical (axial) symmetry. Looking for the solution of Laplace's equation which depends on ρ but not z or ψ (of the cylindrical coordinates (ρ, ψ, z)), we have:

$$\frac{1}{r} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varphi}{\partial \rho}\right) = 0$$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varphi}{\partial \rho}\right) = 0$$

$$\rho \frac{d\varphi}{d\rho} = C$$

$$\frac{d\varphi}{d\rho} = \frac{C}{\rho}$$

$$\varphi = C \ln \rho + D,$$

where C and D are arbitrary constants.

For such φ, the potential difference between the inner and out cylinders is

$$\varphi(a) - \varphi(b) = C \ln a + D - (C \ln b + D)$$

$$= C \ln \frac{a}{b}$$

Setting this equal to V,

$$C \ln \frac{a}{b} = V \Rightarrow C = \frac{V}{\ln \frac{a}{b}}.$$

Since $a<b$, $\ln\frac{a}{b}<0$. so we better write

$$C=-\frac{V}{\ln\frac{b}{a}}$$

The potential then is

$$\varphi=-\frac{V}{\ln b/a}\ln\rho+D.$$

The corresponding electric field is in the ρ direction ($\perp$ to the z axis):

$$\bar{E}=-\vec{\nabla}\varphi=-\frac{\partial\varphi}{\partial\rho}\hat{\rho}$$

$$=+\frac{V}{\ln b/a}\frac{1}{\rho}\hat{\rho}.$$

The current density then is

$$\bar{j}=\sigma\bar{E}=+\frac{\sigma V}{\ln b/a}\frac{1}{\rho}\hat{\rho}.$$

The current that flows from the inner electrode to the outer electrode can be found by integrating $\vec{j}$ over the cylindrical surface of radius ρ, such that $a<\rho<b$:

$$I=\int_S\vec{j}\cdot d\vec{S}=\frac{\sigma V}{\ln b/a}\int_S\frac{1}{\rho}\hat{\rho}\cdot\hat{\rho}\,dS$$

$$=\frac{\sigma V}{\ln b/a}\int_S\frac{1}{\rho}\,dS\underset{\substack{\uparrow\\ \text{since }\rho=\text{const}\\ \text{on the cylindrical surface}}}{=}\frac{\sigma V}{\ln b/a}\frac{1}{\rho}\underbrace{2\pi\rho l}_{\text{area of the cylindr. surface}}=\frac{2\pi\sigma l V}{\ln b/a}$$

From the definition of resistance,

$$R = \frac{V}{I} = \frac{V \ln b/a}{2\pi\sigma l V} = \underline{\underline{\frac{1}{2\pi\sigma l} \ln b/a}}, \quad \text{as required.}$$

(b) For $b - a = d \ll a$, we can transform $\ln b/a$ as

$$\ln\frac{b}{a} = \ln\frac{b-a+a}{a} = \ln\left(1 + \frac{b-a}{a}\right)$$

$$= \ln\left(1 + \frac{d}{a}\right) \simeq \frac{d}{a} \quad \text{for } \frac{d}{a} \ll 1.$$

Hence, the resistance obtained in part (a) becomes

$$R \simeq \frac{1}{2\pi\sigma l}\cdot\frac{d}{a} = \frac{d}{\sigma(2\pi a l)} = \frac{d}{\sigma A},$$

where A is the surface area of the cylinder of radius a and length l (not counting the bases, through which there is no flow of current)

6 (a) Consider the rate of change of the charge Q enclosed by an arbitrary surface S with volume V inside:

$$Q = \int_V \rho(\vec{r}, t)\, dV,$$

where $\rho(\vec{r}, t)$ is the charge density, hence,

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho(\vec{r}, t)}{\partial t} dV.$$

This quantity is the negative of the current that flows across the surface S, so

$$\int_V \frac{\partial \rho}{\partial t} dV = - \oint_S \vec{j} \cdot d\vec{S} .$$ } $\vec{j}$ is the current density here.

Transforming the integral on the right-hand side with the help of Gauss's theorem, we have:

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_V \vec{\nabla} \cdot \vec{j} \, dV ,$$

or
$$\int_V \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} \right) dV = 0 .$$

Since the volume V is arbitrary, we must have

$$\underline{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0} ,$$

which is the continuity equation which expresses the conservation of electric charge.

(b) For the <u>steady</u> current flow, there is no dependence on time for the quantities involved, hence $\frac{\partial \rho}{\partial t} = 0$ and we have from the continuity equation:

$$\vec{\nabla} \cdot \vec{j} = 0 .$$

Using Ohm's law

$$\vec{j} = \sigma \vec{E}$$

and the relation between $\vec{E}$ and φ: $\vec{E} = -\vec{\nabla}\varphi$,

we have from (1) and (2):

$$\vec{\nabla} \cdot (\sigma \vec{E}) = 0$$

$$\Rightarrow \quad \vec{\nabla} \cdot (\sigma \vec{\nabla} \varphi) = 0 ,$$ and if the medium

is homogeneous ($\sigma = \text{const}$), we have

$$\vec{\nabla}\cdot\vec{\nabla}\varphi = 0 \quad \Longleftrightarrow \quad \underline{\underline{\nabla^2\varphi = 0}}\,, \quad \text{i.e.}$$

Laplace's equation.

(c) Using the expression for the Laplacian in cylindrical coordinates, we have:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\psi^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 . \qquad (*)$$

For the potential that is independent of ρ and z, i.e., $\varphi = \varphi(\psi)$, we have from $(*)$:

$$\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\psi^2} = 0 ,$$

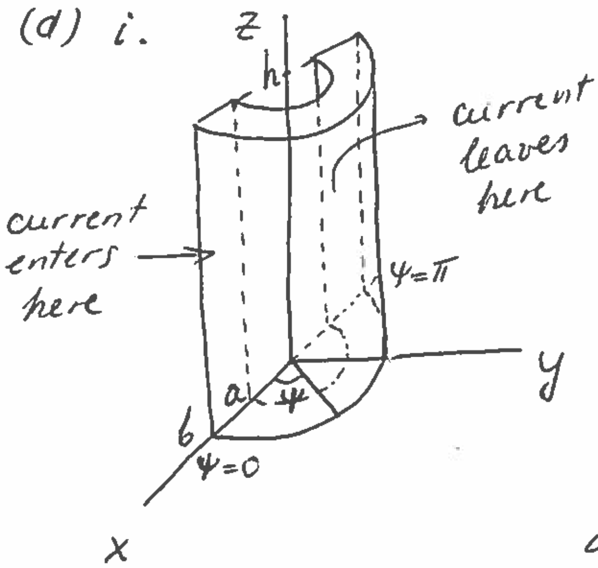
$$\frac{d^2\varphi}{d\psi^2} = 0 ,$$

$$\frac{d\varphi}{d\psi} = C ,$$

$$\varphi = C\psi + D ,$$

where C and D are arbitrary constants.

(d) i.



Using $\varphi = C\psi + D$,

$$\varphi(0) = D = \varphi_1$$

$$\varphi(\pi) = C\pi + D = \varphi_2$$

$$\Rightarrow \quad C\pi + \varphi_1 = \varphi_2$$

$$\Rightarrow \quad C = -\frac{\varphi_1 - \varphi_2}{\pi}$$

and $\varphi = -\dfrac{\varphi_1 - \varphi_2}{\pi}\psi + \varphi_1 = -\dfrac{V}{\pi}\psi + \varphi_1$,

where $V = \varphi_1 - \varphi_2$ is the potential difference.

Using the expression for the gradient in cylindrical coordinates,

$$\vec{\nabla}\varphi = \frac{\partial \varphi}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \psi}\hat{\psi} + \frac{\partial \varphi}{\partial z}\vec{k},$$

we find the electric field

$$\vec{E} = -\vec{\nabla}\varphi = -\frac{1}{\rho}\frac{\partial \varphi}{\partial \psi}\hat{\psi}$$

$$= -\frac{1}{\rho}\frac{d}{d\psi}\left(-\frac{V}{\pi}\psi + \varphi_1\right)\hat{\psi}$$

$$\Rightarrow \quad \vec{E} = \frac{1}{\rho}\frac{V}{\pi}\hat{\psi}.$$

The corresponding current density is

$$\vec{j} = \sigma\vec{E} = \frac{\sigma V}{\pi}\frac{1}{\rho}\hat{\psi}.$$

The current flows along the direction of the unit vector $\hat{\psi}$, i.e., around the z axis, normal to the planes $\psi = const$, such as the electrodes.
The current has no $\hat{\rho}$ or $\vec{k}$ components, so does not cross the cylindrical surfaces or the bases.
The current is found by integrating $\vec{j}$ over a surface of $\psi = const$, for any $0 < \psi < \pi$.

$$I = \int_S \vec{j}\cdot d\vec{S} = \int_S \frac{\sigma V}{\pi}\frac{1}{\rho}\underbrace{\hat{\psi}\cdot\hat{\psi}}_{=1}\, d\rho\, dz$$

$$= \frac{\sigma V}{\pi}\int_0^h\int_a^b \frac{1}{\rho}\, d\rho\, dz = \frac{\sigma V}{\pi}\int_0^h dz\int_a^b \frac{1}{\rho}\, d\rho$$

$$= \frac{\sigma V}{\pi}[z]_0^h\,[\ln\rho]_a^b = \frac{\sigma V}{\pi}h(\ln b - \ln a)$$

$$\Rightarrow \quad \underline{\underline{I = \frac{\sigma h V}{\pi}\ln\frac{b}{a}}}.$$

The resistivity then is

$$R = \frac{V}{I} = \frac{V}{\frac{\sigma h V}{\pi} \ln \frac{b}{a}} = \underline{\underline{\frac{\pi}{\sigma h \ln \frac{b}{a}}}} .$$

For $a, b \to \infty$ with $b-a$ fixed, we use:

$$\ln \frac{b}{a} = \ln \frac{b-a+a}{a} = \ln\left(1 + \frac{b-a}{a}\right) \simeq \frac{b-a}{a},$$

since $\frac{b-a}{a} \ll 1$.

[Here we use the Maclaurin expansion to first order:
$\ln(1+x) \simeq x$.]

~~T~~ Using this in the formula for R, we have:

$$R \simeq \underline{\underline{\frac{\pi a}{\sigma h (b-a)}}} = \frac{l}{\sigma A},$$

where $A = h(b-a)$ is the area of the cross section of the body by the $\varphi = \text{const}$ plane and $l = \pi a$ is the "length" of the resistor, measured as the length of half-circle of radius a.