Electromagnetic Theory AMA3001 **Problem sheet 5**

Magnetic field of steady currents.

Homework

1. Show that the magnetic induction at the centre of a square coil of side $2a$ lying in the $z = 0$ plane and carrying a current I is
$$\boldsymbol{B} = \frac{\sqrt{2}\mu_0 I}{\pi a}\boldsymbol{k}.$$

2. An infinite straight wire whose cross section is a circle of radius a, carries a uniform current I.

   (a) Using cylindrical polar coordinates (ρ, ψ, z), where the z axis coincides with the axis of the wire, show that the magnetic induction is given by
$$\boldsymbol{B} = \frac{\mu_0 I \rho}{2\pi a^2}\hat{\boldsymbol{\psi}} \quad \text{for} \quad \rho \le a, \qquad \boldsymbol{B} = \frac{\mu_0 I}{2\pi \rho}\hat{\boldsymbol{\psi}} \quad \text{for} \quad \rho > a.$$

   (b) Show that this magnetic field can be described by a vector potential of the form $\boldsymbol{A} = A_z(\rho)\boldsymbol{k}$, where
$$A_z(\rho) = -\frac{\mu_0 I \rho^2}{4\pi a^2} \quad \text{for} \quad \rho \le a, \qquad A_z(\rho) = -\frac{\mu_0 I}{4\pi}\left(1 + 2\ln\frac{\rho}{a}\right) \quad \text{for} \quad \rho > a.$$

3. A plane circular loop of radius a lies in the x-y plane with its centre at the origin. A current I flows in the loop.

   (a) By applying the Biot-Savart law, show that the magnetic induction $\boldsymbol{B}$ on the axis of the loop is given by
$$\boldsymbol{B} = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}}\boldsymbol{k}.$$

   (b) Show that in terms of cylindrical polar coordinates (ρ, ψ, z), the vector potential $\boldsymbol{A}$ is given by
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{-a\sin\psi'\,\boldsymbol{i} + a\cos\psi'\,\boldsymbol{j}}{\sqrt{\rho^2 + a^2 + z^2 - 2a\rho\cos(\psi' - \psi)}}\,d\psi'.$$

   (c) Show that this answer can be written in the following form:
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{\pi k}\sqrt{\frac{a}{\rho}}\left[\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right]\hat{\boldsymbol{\psi}},$$
   where
$$k^2 = \frac{4a\rho}{(a+\rho)^2 + z^2},$$
   and $K(k)$ and $E(k)$ are the elliptic integrals,
$$K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}} \quad \text{and} \quad E(k) = \int_0^{\pi/2}\sqrt{1 - k^2\sin^2\alpha}\,d\alpha\,.$$

   (d) Show that near the axis of the loop,
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I a^2}{4}\frac{\rho}{[(a+\rho)^2 + z^2]^{3/2}}\hat{\boldsymbol{\psi}}.$$

4. A solenoid has length L and radius a, and contains N turns of uniformly wound wire. If a current I flows through the wire, show that the magnetic induction on the axis is given by

$$B = \frac{\mu_0 N I}{2L}\left[\frac{L/2 - z}{\sqrt{(L/2-z)^2 + a^2}} + \frac{L/2 + z}{\sqrt{(L/2+z)^2 + a^2}}\right],$$

where the axis is taken to be the z-axis with the point $z = 0$ at the centre of the solenoid.

5. (a) State the Biot-Savart law for the magnetic induction $\boldsymbol{B}$ produced by a steady current distribution $\boldsymbol{j}$.

   (b) Starting from the Biot-Savart law, prove that

$$\nabla \cdot \boldsymbol{B} = 0.$$

   (c) An infinite conducting strip of width $2a$ lies in the y-z plane between $y = -a$ and $y = a$. A uniform current I flows along the strip in the z direction. Find the magnetic induction produced by this current.
   You may assume that

$$\int_{-\infty}^{\infty} \frac{dz}{(\alpha^2 + z^2)^{3/2}} = \frac{2}{\alpha^2}.$$

[Exam 2005 Question 5]

Further Examples

1. (a) State the Biot-Savart law for the magnetic induction $\boldsymbol{B}$ produced by a steady current distribution $\boldsymbol{j}$. Starting from this law, prove that

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}.$$

   (b) A plane circular loop of radius a lies in the x-y plane with its centre at the origin. A current I flows in the loop. By applying Biot-Savart's law, find the magnetic induction $\boldsymbol{B}$ on the axis of the loop.

[Exam 2004 Question 6]

Answers:

5(c). $B_x = \frac{\mu I}{8\pi a} \ln\left[\frac{(y-a)^2 + x^2}{(y+a)^2 + x^2}\right]$; $B_y = \frac{\mu I}{4\pi a}\left[\arctan\left(\frac{a+y}{x}\right) + \arctan\left(\frac{a-y}{x}\right)\right]$; $B_z = 0$.