Magnetic materials.

Homework

1. (a) Write down the equations of magnetostatics in the presence of magnetic media. If there is no free current and the magnetisation is constant, show that you can define a magnetic scalar potential which satisfies Laplace's equation.

   (b) An infinite circular cylinder of radius a is uniformly magnetised perpendicular to its axis with magnetisation $\boldsymbol{M}$. Find the magnetic scalar potential everywhere.

   [Exam 2004 Question 5]

2. A sphere of radius a has uniform magnetisation $\boldsymbol{M}$.

   (a) By solving Laplace's equation in spherical polar coordinates (r, θ, ψ) show that the corresponding magnetic scalar potential ϕ is[1]

   $$\phi = \frac{M}{3} r \cos\theta \quad \text{(inside the sphere)}, \qquad \phi = \frac{Ma^3}{3} \frac{\cos\theta}{r^2} \quad \text{(outside)}.$$

   (b) Hence show that the magnetic induction $\boldsymbol{B}$ is

   $$\boldsymbol{B} = \frac{2}{3}\mu_0 \boldsymbol{M} \quad \text{(inside the sphere)}, \qquad \boldsymbol{B} = \frac{\mu_0 M}{3} \frac{a^3}{r^3} (2\cos\theta\, \hat{\boldsymbol{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}) \quad \text{(outside)}.$$

3. A small magnet of moment $\boldsymbol{m}$ is at the centre of a spherical shell of magnetic material of permeability μ. The inner and outer radii of the shell are a and b respectively. Show that the field outside the shell is the same as that of a magnet of moment $\boldsymbol{m}'$ where

   $$\boldsymbol{m}' = \frac{9\mu\mu_0 b^3 \boldsymbol{m}}{(\mu_0 + 2\mu)(\mu + 2\mu_0) b^3 - 2(\mu - \mu_0)^2 a^3}.$$

   [You may assume that the Legendre expansion of the magnetic scalar potential involves only the term with $P_1(\cos\theta)$.]

   [Exam 2003 Question 3]

4. A uniform sphere of radius a is composed of linear magnetic material of permeability μ. The sphere is placed in an initially uniform magnetic induction $\boldsymbol{B}_0$ parallel to the z axis.

   (a) By solving Laplace's equation in spherical polar coordinates (r, θ, ψ) show that the resulting magnetic scalar potential ϕ is

   $$\phi = -\frac{1}{\mu_0} B_0 r \cos\theta \, \frac{3\mu_0}{\mu + 2\mu_0} \quad \text{(inside the sphere)},$$
   $$\phi = -\frac{1}{\mu_0} B_0 r \cos\theta \left(1 - \frac{\mu - \mu_0}{\mu + 2\mu_0} \frac{a^3}{r^3}\right) \quad \text{(outside)}.$$

   (b) Hence show that the resulting $\boldsymbol{B}$ field is

   $$\boldsymbol{B} = \frac{3\mu}{\mu + 2\mu_0} \boldsymbol{B}_0 \quad \text{(inside the sphere)},$$
   $$\boldsymbol{B} = \boldsymbol{B}_0 + B_0 \frac{\mu - \mu_0}{\mu + 2\mu_0} \frac{a^3}{r^3} (2\cos\theta\, \hat{\boldsymbol{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}) \quad \text{(outside)}.$$

[1] In this problem sheet we denote the magnetic scalar potential ϕ rather than ϕ_m, for simplicity.

5. A long cylinder of radius a and permeability μ is placed in an initially uniform magnetic field $\boldsymbol{B}_0$ such that the cylinder axis is at right angles to $\boldsymbol{B}_0$.

   (a) By solving Laplace's equation in cylindrical polar coordinates (ρ, ψ, z) show that the resulting magnetic scalar potential ϕ is

$$\phi = -\frac{1}{\mu_0} B_0 \rho \cos\psi \, \frac{2\mu_0}{\mu + \mu_0} \quad \text{(inside the cylinder)},$$
$$\phi = -\frac{1}{\mu_0} B_0 \rho \cos\psi \left(1 - \frac{\mu - \mu_0}{\mu + \mu_0} \frac{a^2}{\rho^2}\right) \quad \text{(outside)}.$$

   (b) Hence show that the resulting $\boldsymbol{B}$ field is

$$\boldsymbol{B} = \frac{2\mu}{\mu + \mu_0} \boldsymbol{B}_0 \quad \text{(inside the cylinder)},$$
$$\boldsymbol{B} = \boldsymbol{B}_0 + B_0 \frac{\mu - \mu_0}{\mu + \mu_0} \frac{a^2}{\rho^2} (\cos\psi \, \hat{\boldsymbol{\rho}} + \sin\psi \, \hat{\boldsymbol{\psi}}) \quad \text{(outside)}.$$

6. (a) The vector potential due to a loop C carrying a current I is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} I \oint_C \frac{d\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|},$$

   where μ_0 is the permeability of free space. Show that for $r \gg r'$ this can be approximated by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3},$$

   where $\boldsymbol{m} = I\boldsymbol{a}$ is the magnetic dipole moment and $\boldsymbol{a}$ is the vector area of the loop C,

$$\boldsymbol{a} = \frac{1}{2} \oint_C \boldsymbol{r}' \times d\boldsymbol{r}'.$$

   (b) A magnetic material is modelled by a magnetic dipole moment per unit volume $\boldsymbol{M}$. Show that the magnetisation of the material can be represented by a volume current density $\boldsymbol{j}_m$ and a surface current density $\boldsymbol{J}_m$ given by

$$\boldsymbol{j}_m = \nabla \times \boldsymbol{M} \quad \text{and} \quad \boldsymbol{J}_m = \boldsymbol{M} \times \boldsymbol{n},$$

   where $\boldsymbol{n}$ is a unit outward normal to the surface.

   (c) i. Derive the equation $\nabla \times \boldsymbol{H} = \boldsymbol{j}$, giving the definition of $\boldsymbol{H}$. You may assume the differential form of the Biot-Savart law.

   ii. Briefly describe how the magnetic properties of materials are classified, giving the relationship between $\boldsymbol{M}$ and $\boldsymbol{H}$.

[Exam 2008 Question 5]

Further Examples

1. A spherical hollow of radius a in a medium of permeability μ is centred upon the origin O. A small magnet of moment $\boldsymbol{m}$ is placed at O and points along the z-axis.

   By solving Laplace's equation in spherical polar coordinates (r, θ, ψ), show that the magnetic scalar potential ϕ is given by

$$\phi = \frac{m\cos\theta}{4\pi r^2}\left[1 + \frac{2(\mu_0 - \mu)}{2\mu + \mu_0}\frac{r^3}{a^3}\right] \quad \text{(inside the sphere)},$$
$$\phi = \frac{m\cos\theta}{4\pi r^2}\frac{3\mu_0}{2\mu + \mu_0} \quad \text{(outside)}.$$

2. (a) Show that the vector potential of a uniformly magnetised solid occupying volume V is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\boldsymbol{M} \times \int_S \frac{d\boldsymbol{S}'}{|\boldsymbol{r} - \boldsymbol{r}'|},$$

   where $\boldsymbol{M}$ is the magnetisation and S is the surface of V.

   (b) If the volume V in part (a) is a sphere of radius a, prove that

$$\int_S \frac{d\boldsymbol{S}'}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{\pi\boldsymbol{r}}{r^3}\int_{|a-r|}^{a+r}(a^2 + r^2 - R^2)dR$$

   where $\boldsymbol{r}$ is measured from the centre of the sphere and $R = |\boldsymbol{r} - \boldsymbol{r}'|$.

   Hence show that the vector potential of the uniformly magnetised sphere is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{3}\boldsymbol{M} \times \boldsymbol{r} \qquad (r \le a),$$
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{3}\frac{a^3}{r^3}\boldsymbol{M} \times \boldsymbol{r} \quad (r > a).$$