Electromagnetic Theory AMA3001 Problem sheet 7

Law of induction. Inductance. Magnetic energy.

Homework

1. A small magnet of moment $\boldsymbol{m}$ can slide along the axis of a circular coil of wire of radius a, resistance R and negligible self-inductance. The direction of $\boldsymbol{m}$ is along the axis.

   (a) Show that when the magnet is at a distance z from the centre of the coil, the flux of $\boldsymbol{B}$ across the coil is
   $$\Phi = \frac{\mu_0 m a^2}{2(a^2+z^2)^{3/2}}.$$

   (b) Deduce that if the magnet is moving with velocity v, the current in the coil is
   $$I = \frac{3\mu_0 m a^2 z v}{2R(a^2+z^2)^{5/2}}.$$

2. Calculate the emf in a circular loop of radius a induced by a spatially uniform magnetic field
   $$\boldsymbol{B} = \boldsymbol{B}_0 \cos\omega t,$$
   if $\boldsymbol{B}_0$ makes an angle θ with the normal of the plane of the loop.

3. A transmission line consists of a straight cylindrical wire of radius a inside a coaxial thin conducting cylinder of radius b. The space between them is filled with a magnetic material with permeability μ. The inner wire has permeability μ_0 and carries a uniform current I.

   (a) Find the magnetic induction B inside the line, and hence determine the self-inductance L per unit length of the line.
   [Hint: consider an annulus of radius r and width dr of the inner wire that carries a fraction $2\pi r dr/\pi a^2$ of the total current. Calculate the flux $\Phi(r)$ of the magnetic field B through a rectangle with sides unity and $b-r$, that lies in the plane through the axis, so that one of the unit sides is at a distance r from the axis and the other at b (i.e., on the outer cylinder). Then find the total flux as $\Phi = \int_0^a \Phi(r) 2\pi r dr/\pi a^2$.]

   (b) Calculate the magnetic energy per unit length of the line using the energy density $\frac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B}$, and verify that this energy is equal to $W = \frac{1}{2}LI^2$. (This is in fact a more straightforward way of finding the self-inductance L.)

4. (a) Derive a formula for the magnetic energy W of N circuits carrying currents I_k ($k = 1, \ldots, N$) in the presence of linear magnetic media, given that the magnetic flux through kth circuit is Φ_k.

   (b) Prove that the force $\boldsymbol{F}$ on any component P of the circuits is given by
   $$\{F_x, F_y, F_z\} = \left\{ \left(\frac{\partial W}{\partial x}\right)_I, \left(\frac{\partial W}{\partial y}\right)_I, \left(\frac{\partial W}{\partial z}\right)_I \right\},$$
   where (x, y, z) are the Cartesian coordinates of P and the subscript I indicates that the currents I_k are to be kept fixed.

   (c) A circular loop of radius a carrying current I, is centred on the origin in the x-y plane. (I is regarded positive when the current flows anticlockwise.) A small magnet of moment $\boldsymbol{m} = m\boldsymbol{k}$ is placed on the axis of the loop at point $(0, 0, z)$. Using the results from part (b) and question 1(a), show that the force on the magnet is
   $$F_z = -\frac{3\mu_0 m a^2 I z}{2(a^2+z^2)^{5/2}}.$$

5. An infinite straight wire and a circular loop of radius a lie the same plane, with the centre of the loop being at a distance b ($b > a$) from the straight wire.

(a) Starting from the Biot-Savart law, calculate the magnetic induction $\boldsymbol{B}$ produced by a steady current I flowing in the infinite straight wire.

(b) Using plane polar coordinates for the loop, show that the flux of $\boldsymbol{B}$ through the loop is given by

$$\Phi = \frac{\mu_0 I}{2\pi} \int_0^a dr \int_0^{2\pi} \frac{r\,d\theta}{b + r\cos\theta},$$

and hence, show that the mutual inductance L_{21} between the straight wire and the loop is

$$L_{21} = \mu_0 \left(b - \sqrt{b^2 - a^2} \right). \qquad (1)$$

[Hint: you may use the following integral

$$\int_0^{2\pi} \frac{d\theta}{1 + \alpha\cos\theta} = \frac{2\pi}{\sqrt{1-\alpha^2}} \qquad (|\alpha| < 1),$$

which can be derived using the substitution $t = \tan\frac{\theta}{2}$.]

(c) Verify that for $b \gg a$, the mutual inductance (1) is

$$L_{21} \simeq \frac{\mu_0}{2\pi b} \pi a^2,$$

and explain why this is the correct result.

(d) Find the magnitude of the force between the circular wire and the straight wire when they carry currents I_2 and I_1 respectively.

6. (a) By using the Biot-Savart law to calculate the flux through a circuit C_2 due to a current I_1 in a circuit C_1, show that the mutual inductance L_{21} between the circuits is

$$L_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\boldsymbol{r}_1 \cdot d\boldsymbol{r}_2}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}.$$

This is known as Neumann's formula.

(b) If C_1 and C_2 have self-inductances L_1 and L_2 show that the magnetic energy W of the circuits is

$$W = \frac{1}{2} L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

Hence, using Neumann's formula, prove that the force exerted by C_1 on C_2 is

$$\boldsymbol{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2} \oint_{C_1} \frac{(d\boldsymbol{r}_1 \cdot d\boldsymbol{r}_2)(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}. \qquad (2)$$

(c) Verify that equation (2) is the same as the result

$$\boldsymbol{F}_2 = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2} \oint_{C_1} \frac{d\boldsymbol{r}_2 \times [d\boldsymbol{r}_1 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)]}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3},$$

which can be obtained directly from the Biot-Savart law.

Answers:

2. $\mathcal{E} = \omega\pi a^2 B_0 \cos\theta \sin\omega t$. 3(a). $L = \frac{\mu_0}{8\pi} + \frac{\mu}{2\pi} \ln\frac{b}{a}$.

Further Examples

1. Two equal circular loops of radius a lie opposite each other, a distance b apart.

   (a) Show that the coefficient of mutual inductance is

   $$L_{12} = \frac{\mu_0 a^2}{2} \int_0^{2\pi} \frac{\cos\psi \, d\psi}{\sqrt{b^2 + 2a^2 - 2a^2 \cos\psi}}.$$

   (b) Show that for $b \gg a$ the mutual inductance is given by

   $$L_{12} = \frac{\pi \mu_0 a^4}{2b^3}.$$

   (c) Deduce that if unit currents flow in the same direction round the loops, they attract each other with a force

   $$\frac{3\pi \mu_0 a^4}{2b^4}.$$

2. (a) Obtain a formula for the magnetic energy W of N circuits carrying currents I_k ($k = 1, \ldots, N$) in the presence of linear magnetic media. Deduce that for a volume distribution of current $\boldsymbol{j}$

   $$W = \frac{1}{2} \int \boldsymbol{A} \cdot \boldsymbol{j} \, dV$$

   where $\mathbf{A}$ is the magnetic vector potential.

   (b) An infinitely long wire of radius a carries a uniform current I. Show that a vector potential of the form

   $$\boldsymbol{A} = A_z(\rho)\hat{\boldsymbol{k}},$$

   where ρ is the perpendicular distance from the centre of the wire and $\hat{\boldsymbol{k}}$ is a unit vector parallel to the wire, can be used to describe the associated magnetic field. Find $A_z(\rho)$ both for points inside and outside the wire.

   (c) Two infinitely long wires of radii a and b carry uniform currents I and $-I$, respectively. The centres of the wires are distance h apart. Using the results from parts (a) and (b), show that for $h \gg a, b$ the magnetic energy per unit length of the system is

   $$\frac{\mu_0 I^2}{8\pi} \left(1 + 2 \ln \frac{h^2}{ab}\right).$$