AMA3001 Electromagnetic Theory

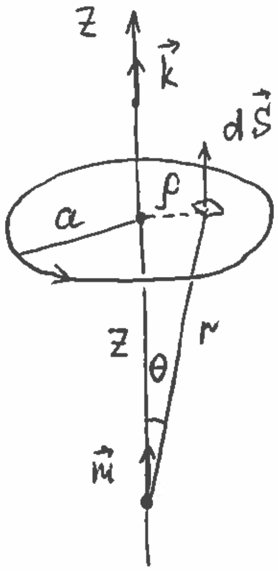
Problem sheet 7

SOLUTIONS

(1) (a) For a magnetic dipole with moment $\vec{m}$, the vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

and the induction is



$$\vec{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m} \right].$$

The flux across the coil is

$$\Phi = \int_S \vec{B} \cdot d\vec{S},$$

where S is the circle of radius a.

To integrate over the circle we use cylindrical coordinates ρ and φ, so that

$$d\vec{S} = \rho\, d\rho\, d\varphi\, \vec{k},$$

$$r = \sqrt{z^2+\rho^2},$$

$$\vec{m}\cdot\vec{r} = mr\cos\theta = mz \quad \text{(see diagram)},$$

$$\vec{r}\cdot\vec{k} = z, \qquad \vec{m}\cdot\vec{k} = m$$

Hence:

$$\int_S \vec{B}\cdot d\vec{S} = \frac{\mu_0}{4\pi} \int_S \frac{1}{(z^2+\rho^2)^{3/2}} \left[\frac{3z^2 m}{z^2+\rho^2} - m \right] \rho\, d\rho\, d\varphi$$

$$= \frac{\mu_0 m}{4\pi} \int_0^{2\pi}\int_0^{a} \left[\frac{3z^2}{(z^2+\rho^2)^{5/2}} - \frac{1}{(z^2+\rho^2)^{3/2}} \right] \rho\, d\rho\, d\varphi$$

Using $\rho\, d\rho = \frac{1}{2} d(\rho^2+z^2)$ we obtain:

$$\frac{\mu_0 m}{4\pi}\,\frac{1}{2}\int_0^{2\pi} d\varphi\left[\int_0^a \frac{3z^2}{(z^2+\rho^2)^{5/2}}\,d(\rho^2+z^2) - \int_0^a \frac{d(\rho^2+z^2)}{(\rho^2+z^2)^{3/2}}\right]$$

$$= \frac{\mu_0 m}{8\pi}\cdot 2\pi\left[3z^2\left(-\frac{2}{3}\,\frac{1}{(z^2+\rho^2)^{3/2}}\right)_0^a + 2\left(\frac{1}{(z^2+\rho^2)^{1/2}}\right)_0^a\right]$$

$$= \frac{\mu_0 m}{4}\left[-2z^2\left(\frac{1}{(z^2+a^2)^{3/2}} - \frac{1}{z^3}\right) + 2\left(\frac{1}{(z^2+a^2)^{1/2}} - \frac{1}{z}\right)\right]$$

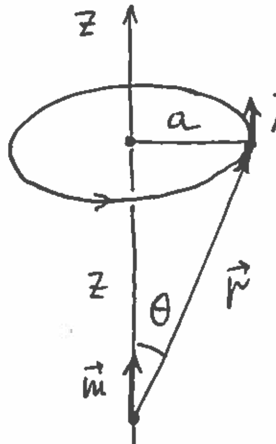
$$= \frac{\mu_0 m}{4}\left[\frac{2}{(z^2+a^2)^{1/2}} - \frac{2z^2}{(z^2+a^2)^{3/2}}\right]$$

$$= \frac{\mu_0 m}{2}\cdot\frac{z^2+a^2-z^2}{(z^2+a^2)^{3/2}} = \frac{\mu_0 m a^2}{2(z^2+a^2)^{3/2}}$$

Hence, the flux is

$$\Phi = \frac{\mu_0 m a^2}{2(a^2+z^2)^{3/2}}.$$

[Note that there is an alternative way of finding the flux from $\Phi = \oint_L \vec{A}\cdot d\vec{r}$, where the integral is over the coil.



$r = (z^2+a^2)^{1/2}$

From $\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$,

the vector $\vec{m}\times\vec{r}$ is tangential to the coil (into the page, for $\vec{r}$ shown).

$|\vec{m}\times\vec{r}| = mr\sin\theta = ma$

$\vec{A}\cdot d\vec{r} = A\,dr$ $\left\{ \begin{array}{l}\vec{A} \text{ and } d\vec{r} \text{ are in} \\ \text{the same direction.}\end{array}\right.$

Hence:

$$\Phi = \oint_L A\,dr = \frac{\mu_0}{4\pi}\oint \frac{ma}{(z^2+a^2)^{3/2}}\,dr$$

$$= \frac{\mu_0 m a\, 2\pi a}{4\pi(z^2+a^2)^{3/2}} = \frac{\mu_0 m a^2}{2(z^2+a^2)^{3/2}}$$ – same result.]

(b) By Faraday's law, the induced emf is

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$= -\frac{d}{dt}\left[\frac{\mu_0 m a^2}{2(z^2+a^2)^{3/2}}\right]$$

$$= +\frac{3\mu_0 m a^2}{4(z^2+a^2)^{5/2}}\, 2z\, \frac{dz}{dt}$$

$$\Rightarrow \quad \mathcal{E} = \frac{3\mu_0 m a^2 z v}{2(z^2+a^2)^{5/2}},$$

where $v = \frac{dz}{dt}$ is the velocity of the magnet.

The current is found from Ohm's law, $IR = V$,

$$I = \frac{\mathcal{E}}{R} = \underline{\underline{\frac{3\mu_0 m a^2 z v}{2R(z^2+a^2)^{5/2}}}}$$

[Note that the current changes sign as the magnet passes through the centre of the coil, as the flux that has increased up to that point, now starts to decrease, causing the sign change in $d\Phi/dt$.]

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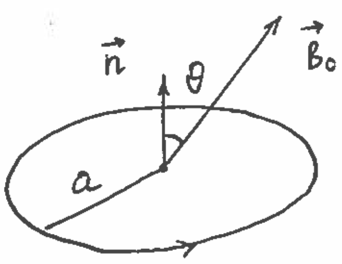


The flux of the magnetic field through the loop is

$$\Phi = \int_S \vec{B}\cdot d\vec{S}$$

$$= \int_S \vec{B}_0 \cos\omega t \cdot \vec{n}\, dS$$

$$= \vec{B}_0\cdot\vec{n}\, \cos\omega t \int dS$$

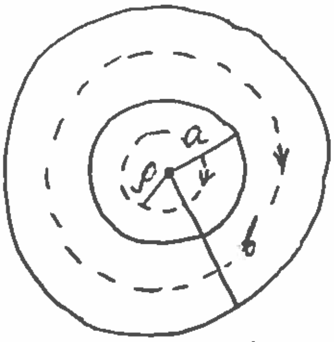
$$= B_0 \cos\theta \cos\omega t \;\; \pi a^2 .$$

The emf then is

$$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}\left(B_0 \cos\theta \cos\omega t \; \pi a^2\right)$$

$$= \pi B_0 a^2 \cos\theta \; \omega \sin\omega t .$$

(3) (a)



[View down the transmission line]

By symmetry, the magnetic field ($\vec{H}$ and $\vec{B}$) depends only on the distance ρ from the axis and is tangential to the circles centred on the axis. Using such circles as lines L in the integral form of Ampere's law

$$\oint_L \vec{H}\cdot d\vec{r} = I_L \quad (\text{current enclosed in } L)$$

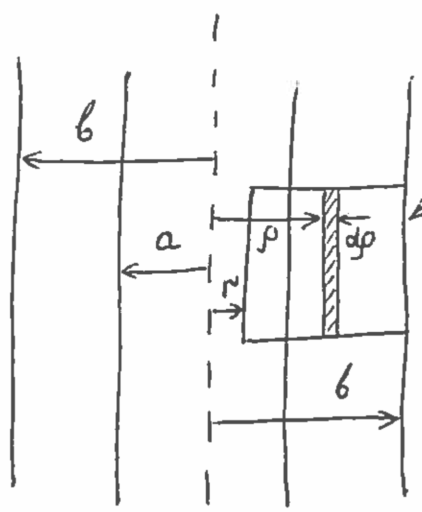
we have:

for $\rho < a$ $\qquad 2\pi\rho H = \dfrac{I \pi \rho^2}{\pi a^2}$

$$\Rightarrow \quad H = \frac{I\rho}{2\pi a^2} \;, \qquad B = \mu_0 H = \frac{\mu_0 I \rho}{2\pi a^2} .$$

for $a < \rho < b$ $\qquad 2\pi\rho H = I$

$$\Rightarrow \quad H = \frac{I}{2\pi\rho} \;, \qquad B = \mu H = \frac{\mu I}{2\pi\rho} .$$



Rectangle for the calculation of the flux.

Cross section through the axis of the line

The flux through the rectangle (of unit length along the line) is

$$\Phi(r) = \int_S \vec{B}\cdot d\vec{S} = \int B \, dS \qquad \left\{ \begin{array}{l} \vec{B} \text{ is in the same} \\ \text{direction as } d\vec{S} \end{array} \right.$$

$$= \int_r^b B(\rho)\, d\rho$$

$dS = 1 \cdot d\rho$ (unit length)

$$\Phi(r) = \int_r^a \frac{\mu_0 I \rho}{2\pi a^2} d\rho + \int_a^b \frac{\mu I}{2\pi\rho} d\rho$$

$$= \frac{\mu_0 I}{2\pi a^2} \int_r^a \rho \, d\rho + \frac{\mu I}{2\pi} \int_a^b \frac{d\rho}{\rho}$$

$$= \frac{\mu_0 I}{4\pi a^2}(a^2 - r^2) + \frac{\mu I}{2\pi} \ln b/a .$$

This flux is associated with the fraction $\frac{2\pi r\,dr}{\pi a^2}$ of the current. The total flux is obtained as

$$\Phi = \int_0^a \Phi(r) \frac{2\pi r\,dr}{\pi a^2}$$

$$= \frac{2}{a^2} \int_0^a \left[\frac{\mu_0 I}{4\pi a^2}(a^2 - r^2) + \frac{\mu I}{2\pi} \ln b/a \right] r\,dr$$

$$= \frac{2}{a^2} \left[\frac{\mu_0 I}{4\pi a^2} \int_0^a (a^2 - r^2) r\,dr + \frac{\mu I}{2\pi} \ln b/a \int_0^a r\,dr \right]$$

$$= \frac{2}{a^2} \left[\frac{\mu_0 I}{4\pi a^2} \left(a^2 \frac{a^2}{2} - \frac{a^4}{4} \right) + \frac{\mu I}{2\pi} \ln b/a \, \frac{a^2}{2} \right]$$

$$= \frac{\mu_0 I}{2\pi a^4} a^4 \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{\mu I}{2\pi} \ln b/a$$

$$= \frac{\mu_0 I}{8\pi} + \frac{\mu I}{2\pi} \ln b/a .$$

The self-inductance is then

$$L = \frac{\Phi}{I} = \frac{\mu_0}{8\pi} + \frac{\mu}{2\pi} \ln \frac{b}{a} .$$

(b) The magnetic energy (per unit length of the line) is:

$$W = \frac{1}{2}\int_V \vec{H}\cdot\vec{B}\, dV = \frac{1}{2}\int_V HB\, 2\pi\rho\, d\rho$$

Integrating over cylindrical shells of unit length

$$= \frac{1}{2}\int_0^a \mu_0 H^2\, 2\pi\rho\, d\rho + \frac{1}{2}\int_a^b \mu H^2\, 2\pi\rho\, d\rho$$

$$= \mu_0 \pi \int_0^a \frac{I^2\rho^2}{4\pi^2 a^4}\rho\, d\rho + \mu\pi \int_a^b \frac{I^2}{4\pi^2\rho^2}\rho\, d\rho$$

$$= \frac{\mu_0 I^2}{4\pi a^4}\int_0^a \rho^3\, d\rho + \frac{\mu I^2}{4\pi}\int_a^b \frac{d\rho}{\rho}$$

$$= \frac{\mu_0 I^2}{4\pi a^4}\cdot\frac{a^4}{4} + \frac{\mu I^2}{4\pi}\ln b/a$$

$$= \frac{1}{2}\left(\frac{\mu_0}{8\pi} + \frac{\mu}{2\pi}\ln b/a\right) I^2$$

Comparing this with $W = \frac{1}{2}LI^2$, we see that

$L = \frac{\mu_0}{8\pi} + \frac{\mu}{2\pi}\ln\frac{b}{a}$ <u>is</u> the self-inductance per unit length of the line.

4 (a) Suppose, the currents are changed from I_k to $I_k + dI_k$ over a small time interval dt, with the associated change $d\Phi_k$ in the flux through kth circuit. The change of the flux causes induced emf

$$\mathcal{E}_k = -\frac{d\Phi_k}{dt}$$

in kth circuit. The work <u>against it</u> to sustain the current in circuit k is

$$dW_k = -\mathcal{E}_k\, dQ_k = \frac{d\Phi_k}{dt}\, dQ_k = d\Phi_k \frac{dQ_k}{dt} = d\Phi_k I_k.$$

The total work done for all the circuits then is

$$dW = \sum_{k=1}^{N} I_k \, d\Phi_k .$$

Let us now assume that the currents are increased from zero to their final values I_k by changing a parameter α from zero to unity, so that the currents are αI_k. Since the fluxes are proportional to the magnetic field, which in turn is proportional to the currents, the fluxe for a given value of α are $\alpha \Phi_k$. The work in changing α from α to $\alpha + d\alpha$ then is

$$dW = \sum_{k=1}^{N} \alpha I_k \, d(\alpha \Phi_k)$$

$$= \sum_{k=1}^{N} I_k \Phi_k \, \alpha \, d\alpha .$$

Hence, the total work on increasing α from 0 to 1 is

$$W = \int dW = \int_0^1 \sum_{k=1}^{N} I_k \Phi_k \, \alpha \, d\alpha$$

$$= \sum_{k=1}^{N} I_k \Phi_k \int_0^1 \alpha \, d\alpha$$

$$\Rightarrow \quad \underline{\underline{W = \frac{1}{2} \sum_{k=1}^{N} I_k \Phi_k .}}$$

The total work done in establishing the currents and magnetic fluxes is the energy of the magnetic field.

(b) Let one of the components of the system be moved through displacement $d\vec{r}$. The work by the force $\vec{F}$ acting on this component is

$$\vec{F} \cdot d\vec{r} = -dW + dW_b ,$$

where $dW = \frac{1}{2} \sum_{k=1}^{N} I_k \, d\Phi_k$ is the change in the magnetic energy and dW_b is the work by the batteries on sustaining the currents I_k.

The work by the batteries is against the emf's

$$\mathcal{E}_k = -\frac{d\Phi_k}{dt}$$

caused by the changing magnetic fluxes.

Hence,

$$dW_b = \sum_{k=1}^{N} -\mathcal{E}_k \, dQ_k$$

dQ_k - charges moved around the circuits

$$= \sum_{k=1}^{N} \frac{d\Phi_k}{dt} dQ_k$$

$$= \sum_{k=1}^{N} d\Phi_k \frac{dQ_k}{dt}$$

$$= \sum_{k=1}^{N} I_k \, d\Phi_k \, .$$

We see that $\quad dW_b = 2dW$, so the work is

$$\vec{F} \cdot d\vec{r} = -dW + 2dW = dW,$$

and the three components of the force are found as:

$$F_x = \left(\frac{\partial W}{\partial x}\right)_I, \quad F_y = \left(\frac{\partial W}{\partial y}\right)_I, \quad F_z = \left(\frac{\partial W}{\partial z}\right)_I,$$

where the subscript I indicates that the derivatives are taken for fixed currents I_k.

(c) The magnet can be considered as a small loop of current with magnetic moment $\vec{m}$. The total magnetic energy of the system can then be written as

$$W = \frac{1}{2} L_1 I^2 + L_{12} I I' + \frac{1}{2} L_2 I'^2, \qquad (*)$$

where L_1 is the self-inductance of the loop, L_2 is the self-inductance of the loop that represents the magnet (with I' being the associated current), and L_{12} being the mutual inductance. By definition,

$$L_{12} I' = \Phi_1$$

- flux through the loop due to the magnet. This flux is the same as Φ found in question 1(a),

$$\Phi = \frac{\mu_0 m a^2}{2(a^2+z^2)^{3/2}} .$$

When the magnet is moved relative to the loop, the first and last terms in (*) remain unchanged, and it is only the middle term that is sensitive to the relative position of the two. This term is given by $I\Phi$, so the force on the magnet is:

$$F_z = \frac{d}{dz}(I\Phi) = \frac{d}{dz}\,\frac{\mu_0 m a^2 I}{2(a^2+z^2)^{3/2}}$$

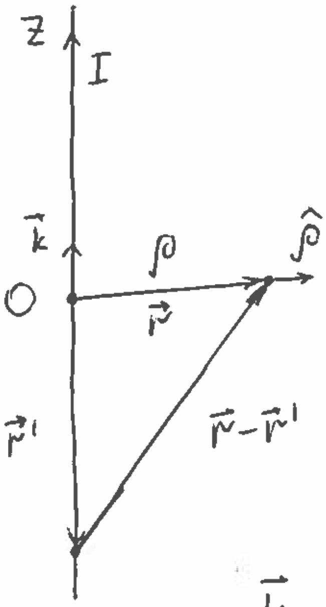
$$= -\frac{3}{2}\,\frac{\mu_0 m a^2 I}{2(a^2+z^2)^{5/2}}\cdot 2z$$

$$= -\frac{3\mu_0 m a^2 I z}{2(a^2+z^2)^{5/2}} .$$

[For $z>0$ $F_z<0$, and for $z<0$ $F_z>0$, so the magnet is always attracted to the loop.]

(5) (a) Biot-Savart's law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_C \frac{d\vec{r}\,'\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$



Let $\vec{r} = \rho\hat{\rho}$

$\vec{r}\,' = z\vec{k}$ (wire along the z axis)

Then $d\vec{r}\,' = \vec{k}\,dz$

$$d\vec{r}\,'\times(\vec{r}-\vec{r}\,') = dz\;\vec{k}\times(\rho\hat{\rho}-\vec{k}z)$$

$$= \vec{k}\times\hat{\rho}\,\rho\,dz = \hat{\varphi}\,\rho\,dz$$

$\vec{k}\times\hat{\rho} = \hat{\varphi}$ (unit vector, into the page).
(in general, $\perp$ to z axis and to radius ρ).

$$|\vec{r}-\vec{r}'| = |\rho\hat{\rho} - z\vec{k}| = \sqrt{\rho^2+z^2}\ .$$

Therefore :

$$\vec{B}(\rho) = \frac{\mu_0 I}{4\pi}\hat{\varphi}\int_{-\infty}^{+\infty} \frac{\rho\, dz}{(\rho^2+z^2)^{3/2}}$$

Let $\quad z = \rho \tan\alpha$,

$$dz = \rho\frac{d\alpha}{\cos^2\alpha}\ ,$$

$$\rho^2+z^2 = \rho^2(1+\tan^2\alpha) = \frac{\rho^2}{\cos^2\alpha}\ ,$$

limits : $\quad z = -\infty$ corresponds to $\alpha = -\frac{\pi}{2}$

$\quad z = +\infty$ corresponds to $\alpha = \frac{\pi}{2}$.

Hence:

$$\vec{B}(\rho) = \frac{\mu_0 I}{4\pi}\hat{\varphi}\int_{-\pi/2}^{\pi/2} \frac{\rho^2\, d\alpha\, \cos^3\alpha}{\cos^2\alpha\ \rho^3}$$

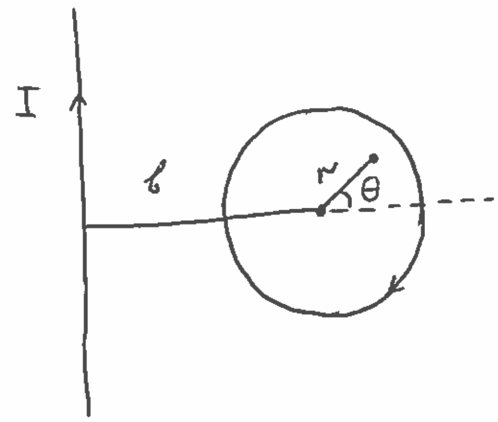
$$= \frac{\mu_0 I}{4\pi\rho}\hat{\varphi}\int_{-\pi/2}^{\pi/2} \cos\alpha\, d\alpha$$

$$= \frac{\mu_0 I}{4\pi\rho}\Big[\sin\alpha\Big]_{-\pi/2}^{\pi/2}\hat{\varphi}$$

$$= \frac{\mu_0 I}{4\pi\rho}\big[1-(-1)\big]\hat{\varphi}$$

$$= \frac{\mu_0 I}{2\pi\rho}\hat{\varphi}\ .$$

(b)



We introduce polar coordinates (r, θ) on the loop (see diagram). The distance to the wire is

$$\rho = b + r\cos\theta$$

The magnetic field at this point is (from part (a))

$$\vec{B} = \frac{\mu_0 I}{2\pi(b + r\cos\theta)}\hat{\varphi}.$$

The element of area is $dS = r\,dr\,d\theta$ and we take the normal to the loop along $\hat{\varphi}$ (i.e. into the page plane). The flux through the loop is then given by

$$\Phi = \int_S \vec{B}\cdot d\vec{S} = \frac{\mu_0 I}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{r\,dr\,d\theta}{b + r\cos\theta}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^a dr\int_0^{2\pi}\frac{r\,d\theta}{b + r\cos\theta}.$$

Hence,

$$\Phi = \frac{\mu_0 I}{2\pi}\int_0^a r\,dr\int_0^{2\pi}\frac{d\theta}{b\left(1 + \frac{r}{b}\cos\theta\right)}$$

$$= \frac{\mu_0 I}{2\pi b}\int_0^a r\,dr\,\frac{2\pi}{\sqrt{1 - \frac{r^2}{b^2}}} \quad \left\{ \text{Using } \int_0^{2\pi}\frac{d\theta}{1 + \alpha\cos\theta} = \frac{2\pi}{\sqrt{1-\alpha^2}} \right.$$

$$= \mu_0 I\int_0^a\frac{r\,dr}{\sqrt{b^2 - r^2}}$$

$$= -\frac{\mu_0 I}{2}\int_0^a\frac{d(b^2 - r^2)}{\sqrt{b^2 - r^2}} = -\frac{\mu_0 I}{2}\left[2\sqrt{b^2 - r^2}\right]_0^a$$

$$= -\mu_0 I\left[\sqrt{b^2 - a^2} - b\right]$$

Hence, the flux is

$$\Phi = \mu_0 I \left(b - \sqrt{b^2 - a^2} \right)$$

and the mutual inductance

$$L_{21} = \frac{\Phi}{I} = \underline{\underline{\mu_0 \left(b - \sqrt{b^2 - a^2} \right)}}.$$

(c) For $b \gg a$, we have

$$\sqrt{b^2 - a^2} = b \sqrt{1 - \frac{a^2}{b^2}} = b \left(1 - \frac{a^2}{b^2} \right)^{1/2}$$

$$\simeq b \left(1 - \frac{1}{2} \frac{a^2}{b^2} \right)$$

$$= b - \frac{a^2}{2b}$$

Using binomial expansion: $(1+x)^\alpha \simeq 1 + \alpha x + \ldots$

Hence,

$$L_{21} \simeq \mu_0 \left(b - b + \frac{a^2}{2b} \right) = \frac{\mu_0 a^2}{2b}$$

$$= \frac{\mu_0}{2\pi b} \pi a^2$$

For $b \gg a$, the magnetic field at the loop is approximately constant and given by $B = \frac{\mu_0 I}{2\pi b}$, and its flux is

$$\Phi \simeq B \pi a^2 = \frac{\mu_0 I}{2\pi b} \pi a^2,$$

so that the mutual inductance is

$$L_{21} = \frac{\Phi}{I} \simeq \frac{\mu_0}{2\pi b} \pi a^2, \quad \text{as above.}$$

(d) The magnetic energy is $W = \frac{1}{2} L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2} L_2 I_2^2$ and the part that depends on the mutual position of the loop and wire is $L_{21} I_1 I_2$

Hence, the force is found as

$$F = \frac{\partial W}{\partial b} = \frac{\partial}{\partial b} \left(L_{21} I_1 I_2 \right)$$

This force is in the ρ direction, $\perp$ to the wire

Using the expression from part (b), we have:

$$F = \frac{\partial}{\partial b}\left[\mu_0\left(b - \sqrt{b^2-a^2}\right) I_1 I_2\right]$$

$$= \mu_0 I_1 I_2 \left(1 - \frac{2b}{2\sqrt{b^2-a^2}}\right)$$

$$= \mu_0 I_1 I_2 \left(1 - \frac{b}{\sqrt{b^2-a^2}}\right).$$

Note that this force is negative (since $\frac{b}{\sqrt{b^2-a^2}} > 1$), so the loop is attracted to the wire. The magnitude of the force then is

$$|F| = \mu_0 I_1 I_2 \left(\frac{b}{\sqrt{b^2-a^2}} - 1\right).$$

[The force is attractive if the current in the loop on the diagram (page 11) is clockwise, which is the positive direction with respect to the normal $\hat{\varphi}$ directed into the page.]

6 (a) Let $\vec{B}_1$ be the magnetic field created by circuit C_1 carrying current I_1. The flux of this magnetic field through C_2 is

$$\Phi_2 = \int_{S_2} \vec{B}_1 \times d\vec{S} = \oint_{C_2} (\vec{\nabla}\times\vec{B}_1)\cdot d\vec{r}_2$$

$$= \oint_{C_2} \vec{A}_1 \cdot d\vec{r}_2, \qquad (1)$$

when $\vec{A}_1$ is the vector potential due to the current I_1 in C_1. At point $\vec{r}_2$ this potential is

$$A_1(\vec{r}_2) = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \frac{d\vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}. \qquad (2)$$

Substituting (2) into (1) we find

$$\Phi_2 = \frac{\mu_0}{4\pi} I_1 \oint_{C_2} \oint_{C_1} \frac{d\vec{r}_1 \cdot d\vec{r}_2}{|\vec{r}_2 - \vec{r}_1|} ,$$

and the mutual inductance is

$$L_{21} = \frac{\Phi_2}{I_1} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{r}_1 \cdot d\vec{r}_2}{|\vec{r}_2 - \vec{r}_1|} .$$

(b) The magnetic energy of the system of two circuits can be written as

$$W = \frac{1}{2} (\Phi_1 I_1 + \Phi_2 I_2) \qquad (3)$$

Here

$$\Phi_1 = L_1 I_1 + L_{12} I_2 ,$$

$$\Phi_2 = L_2 I_2 + L_{21} I_1 ,$$

where L_1 and L_2 are the self-inductances of circuits C_1 and C_2. Substituting Φ_1 and Φ_2 into (3) gives

$$W = \frac{1}{2} \left(L_1 I_1^2 + L_{12} I_1 I_2 + L_{21} I_1 I_2 + L_2 I_2^2 \right)$$

$$= \frac{1}{2} L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2} L_2 I_2^2 ,$$

where we used $L_{12} = L_{21}$ (seen explicitly in the result from part (a), which is symmetric with respect to interchanging $1 \rightleftarrows 2$).

If circuit C_2 is displaced by a small vector $\vec{a}$, then $\vec{r}_2 \to \vec{r}_2 + \vec{a}$, and the associated work can be related to the change in W by

$$\vec{F}_2 \cdot \vec{a} = \delta W .$$

The change in W is due to the change in L_{21}, so we have:

$$\delta W = I_1 I_2 \, \delta L_{21} ,$$

where

$$\delta L_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{r}_1 \cdot d\vec{r}_2}{|\vec{r}_2 + \vec{a} - \vec{r}_1|} - \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{r}_1 \cdot d\vec{r}_2}{|\vec{r}_2 - \vec{r}_1|} . \qquad (*)$$

Expanding to first order in $\vec{a}$,

$$\frac{1}{|\vec{r}_2 + \vec{a} - \vec{r}_1|} = \frac{1}{|\vec{r}_2 - \vec{r}_1|} + \left(\vec{\nabla}_2 \frac{1}{|\vec{r}_2 - \vec{r}_1|}\right) \cdot \vec{a} + \ldots$$

$$\simeq \frac{1}{|\vec{r}_2 - \vec{r}_1|} - \frac{(\vec{r}_2 - \vec{r}_1) \cdot \vec{a}}{|\vec{r}_2 - \vec{r}_1|^3}$$

Substituting this into $(*)$, we find

$$\delta L_{21} = - \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{r}_1 \cdot d\vec{r}_2 \, (\vec{r}_2 - \vec{r}_1) \cdot \vec{a}}{|\vec{r}_2 - \vec{r}_1|^3}$$

and

$$\delta W = - \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2} \oint_{C_1} \frac{d\vec{r}_1 \cdot d\vec{r}_2 \, (\vec{r}_2 - \vec{r}_1) \cdot \vec{a}}{|\vec{r}_2 - \vec{r}_1|^3} = \vec{F}_2 \cdot \vec{a} ,$$

hence

$$\vec{F}_2 = - \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2} \oint_{C_1} \frac{d\vec{r}_1 \cdot d\vec{r}_2 \, (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} .$$

(c) The force acting on C_2 due to the magnetic field created by C_1 can be calculated from Biot-Savart's formula as

$$\vec{F}_2 = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2} \oint_{C_1} \frac{d\vec{r}_2 \times [d\vec{r}_1 \times (\vec{r}_2 - \vec{r}_1)]}{|\vec{r}_2 - \vec{r}_1|^3} .$$

Let us show that this expression is in fact equivalent to that obtained in part (b).

Using "bac - cab" rule,

$$\underset{a}{d\vec{r}_2} \times [\underset{b}{d\vec{r}_1} \times \underset{c}{(\vec{r}_2 - \vec{r}_1)}] = d\vec{r}_1 \left(d\vec{r}_2 \cdot (\vec{r}_2 - \vec{r}_1) \right)$$

$$- (\vec{r}_2 - \vec{r}_1)\,(d\vec{r}_1 \cdot d\vec{r}_2)$$

Substituting into $\vec{F}_2$:

$$\vec{F}_2 = \frac{\mu_0}{4\pi} I_1 I_2 \left[\underbrace{\oint_{C_2}\oint_{C_1} \frac{d\vec{r}_1 \left(d\vec{r}_2 \cdot (\vec{r}_2 - \vec{r}_1)\right)}{|\vec{r}_2 - \vec{r}_1|^3}} - \oint_{C_2}\oint_{C_1} \frac{(\vec{r}_2 - \vec{r}_1)(d\vec{r}_1 \cdot d\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|^3} \right]$$

$$\oint_{C_1} d\vec{r}_1 \oint_{C_2} \frac{d\vec{r}_2 \cdot (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

$$= - \oint_{C_1} d\vec{r}_1 \oint_{C_2} d\vec{r}_2 \cdot \vec{\nabla}_2 \frac{1}{|\vec{r}_2 - \vec{r}_1|}$$

$$= 0 \qquad \left(\text{as any } \oint \vec{\nabla} f \cdot d\vec{r} \right).$$

Hence, only the 2nd term is square brackets contributes, and we have

$$\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2}\oint_{C_1} \frac{(d\vec{r}_1 \cdot d\vec{r}_2)(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3},$$

which is exactly the same answer as obtained in part (b).