# AMA3001 Electromagnetic Theory

## Problem sheet 8

## SOLUTIONS

① (a) Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 .$$

The current density is given by Ohm's law:

$$\vec{j} = \sigma \vec{E} ,$$

so we have:

$$\frac{\partial \rho}{\partial t} + \sigma \vec{\nabla} \cdot \vec{E} = 0 \quad \Big\} \begin{array}{l} \text{since } \sigma = \text{const} \\ \vec{\nabla} \cdot (\sigma \vec{E}) = \sigma \vec{\nabla} \cdot \vec{E} \end{array}$$

From Maxwell's first equation,

$$\vec{\nabla} \cdot \vec{D} = \rho ,$$

using $\vec{D} = \varepsilon \vec{E}$ with $\varepsilon = \text{const}$,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon} ,$$

so we obtain:

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon} \rho = 0 . \qquad (*)$$

At a fixed point $\vec{r}$ in space, $\rho$ depends only on time and $(*)$ can be written as

$$\frac{d\rho}{dt} = -\frac{\sigma}{\varepsilon} \rho \qquad \text{(ordinary differential eqn.)}$$

$$\int \frac{d\rho}{\rho} = -\frac{\sigma}{\varepsilon} \int dt$$

$$\ln \rho = -\frac{\sigma}{\varepsilon} t + C$$

Denoting $\varepsilon / \sigma \equiv \tau$, we have:

$$\rho = e^{-t/\tau} \underbrace{e^{c}}_{\text{rename } C}.$$

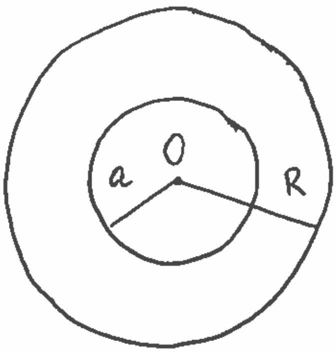
$$\rho = C e^{-t/\tau}.$$

At a given point $\bar{r}$, at $t=0$, $\rho = \rho_0(\bar{r})$

$\Rightarrow$ $C = \rho_0(\bar{r})$ and the charge density at time $t$ is given by:

$$\underline{\underline{\rho(\bar{r},t) = \rho_0(\bar{r})\, e^{-t/\tau}}}.$$

(b) i.




Integrating the relation obtained in part (a) over any fixed volume $V$ we obtain:

$$\int_V \rho(\bar{r},t)\,dV = e^{-t/\tau} \int_V \rho_0(\bar{r})\,dV$$

or

$$Q(t) = Q_0 e^{-t/\tau}, \qquad (*)$$

where $Q_0$ is the charge inside $V$ at $t=0$ and $Q(t)$ is the charge inside $V$ at time $t$.

Choosing $V$ as a thin spherical shell which encloses the sphere of radius $a$, we see that $(*)$ describes the time dependence of the charge on this sphere.

The charge density corresponding to the initial charge distribution can be written using the delta function as $\rho_0(\bar{r}) = \dfrac{Q_0}{4\pi a^2}\,\delta(r-a)$, which can be verified by integrating $\rho_0(\bar{r})$ over any spherical shell with inner radius smaller than $a$ and outer radius greater than $a$.

The charge density everywhere else inside the sphere (i.e., $r < R$ and $r \neq a$) was zero at $t = 0$; hence it remains zero.

The charges from $r = a$ migrate to the surface $r = R$. (They "want" to be as far from each other as possible.)

Since the total charge is conserved, the $\overbrace{\text{charge}}^{\text{surface}}$ on the sphere $r = R$ is $Q_0 - Q e^{-t/\tau} = Q_0 (1 - e^{-t/\tau})$.

ii. To find the amount of energy dissipated over a time interval, we need to find the <u>rate</u>

$$\int_V \vec{j} \cdot \vec{E} \, dV$$

at which the energy is dissipated.

Due to the symmetry of the system, the electric field is zero for $r < a$. For $a < r < R$ it can be found from Gauss's law:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon},$$

using a spherical surface of radius $r$. Since $\vec{E}$ is in the radial direction, we have:

$$E \cdot 4\pi r^2 = \frac{Q_0}{\varepsilon} e^{-t/\tau}$$

$$\Rightarrow \quad \vec{E} = \frac{Q_0}{4\pi \varepsilon r^2} e^{-t/\tau} \hat{r}.$$

The current density is

$$\vec{j} = \sigma \vec{E} = \frac{\sigma Q_0}{4\pi \varepsilon r^2} e^{-t/\tau} \hat{r}.$$

The rate:

$$\int_V \vec{j}\cdot\vec{E}\,dV = \int_a^R \frac{\sigma Q_0}{4\pi\varepsilon r^2} e^{-t/\tau} \frac{Q_0}{4\pi\varepsilon r^2} e^{-t/\tau} \underbrace{4\pi r^2 dr}_{dV}$$

[We are integrating over spherical shells]

$$= \frac{\sigma Q_0^2}{4\pi\varepsilon^2} e^{-2t/\tau} \int_a^R \frac{dr}{r^2}$$

$$= \frac{\sigma Q_0^2}{4\pi\varepsilon^2} e^{-2t/\tau} \left(\frac{1}{a} - \frac{1}{R}\right)$$

$$= \frac{\sigma Q_0^2 (R-a)}{4\pi a R \varepsilon^2} e^{-2t/\tau} .$$

The total amount of energy dissipated between times $t=0$ and $t=T$ is

$$\int_0^T \frac{\sigma Q_0^2 (R-a)}{4\pi a R \varepsilon^2} e^{-2t/\tau} dt$$

$$\int e^{-2t/\tau} dt = -\frac{\tau}{2} e^{-2t/\tau} + C$$

$$= \frac{\sigma Q_0^2 (R-a)}{4\pi a R \varepsilon^2} \frac{\tau}{2} \left[-e^{-2t/\tau}\right]_0^T$$

$$= \frac{\sigma Q_0^2 (R-a)\tau}{8\pi a R \varepsilon^2} \left[1 - e^{-2T/\tau}\right]$$

Using $\tau = \frac{\varepsilon}{\sigma}$

$$= \frac{Q_0^2 (R-a)}{8\pi a R \varepsilon} \left(1 - e^{-2T/\tau}\right) .$$

The electric field outside the sphere $r=R$ does not change, because the total charge of the sphere is $Q_0$ at all times (and the field is $E = \frac{Q_0}{4\pi\varepsilon_0 r^2}$, assuming the sphere is in vacuum, or medium with permittivity $\varepsilon_0$).

The energy of the field between $r=a$ and $r=R$ is

$$\int_V w\,dV = \frac{1}{2}\int_V \vec{E}\cdot\vec{D}\,dV$$

$$= \frac{1}{2}\int_a^R \varepsilon E^2\, 4\pi r^2 dr \qquad \Big\}\ \text{Using } E \text{ from page 3}$$

$$= \frac{1}{2}\,\varepsilon\int_a^R \frac{Q_0^2}{16\pi^2\varepsilon^2 r^4}\, e^{-2t/\tau}\, 4\pi r^2 dr$$

$$= \frac{Q_0^2}{8\pi\varepsilon}\int_a^R \frac{dr}{r^2}$$

$$= \frac{Q_0^2}{8\pi\varepsilon}\, e^{-2t/\tau}\left(\frac{1}{a}-\frac{1}{R}\right)$$

$$= \frac{Q_0^2(R-a)}{8\pi a R\varepsilon}\, e^{-2t/\tau}.$$

The change in this energy between times $t=0$ and $t=T$ is

$$\frac{Q_0^2(R-a)}{8\pi a R\varepsilon}\, e^{-2T/\tau} - \frac{Q_0^2(R-a)}{8\pi a R\varepsilon}$$

$$= -\frac{Q_0^2(R-a)}{8\pi a R\varepsilon}\left(1-e^{-2T/\tau}\right),$$

which is the negative of the amount of energy dissipated in the medium due to the currents (see bottom of page 4), as expected from the general relation for the energy of the electromagnetic field

$$\frac{\partial}{\partial t}\int_V w\,dV = -\underbrace{\oint_S \vec{S}\cdot d\vec{S}}_{0 \text{ in this problem}} - \int_V \vec{j}\cdot\vec{E}\,dV$$

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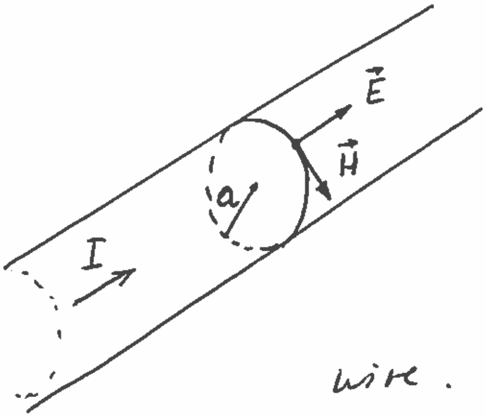



Current density in the wire

$$j = \frac{I}{\pi a^2}$$

The magnetic field H at any point is tangential to the circle centred on the axis of the wire. Its magnitude on the surface of the wire is found from

$$\oint_L \vec{H} \cdot d\vec{r} = I$$

$$2\pi a H = I$$

$$\Rightarrow \qquad H = \frac{I}{2\pi a}.$$

The electric field is along the wire (in the same direction as the current and $\vec{j}$) and, from $\vec{j} = \sigma \vec{E}$, we have:

$$E = \frac{j}{\sigma} = \frac{I}{\pi a^2 \sigma}.$$

The Poynting vector

$$\vec{S} = \vec{E} \times \vec{H}$$

is radially <u>into</u> the wire. Its magnitude is

$$S = EH = \frac{I}{\pi a^2 \sigma} \cdot \frac{I}{2\pi a} = \frac{I^2}{2\pi^2 a^3 \sigma}.$$

The integral of $\vec{S}$ over the surface of wire for a segment of length L is

$$\int \underbrace{\vec{S}}_{\text{inwards}} \cdot \underbrace{d\vec{S}}_{\text{outwards}} = -S\, 2\pi a L = -\frac{I^2\, 2\pi a L}{2\pi^2 a^3 \sigma}$$

$$= -\frac{I^2 L}{\pi a^2 \sigma}.$$

This shows the amount of electromagnetic energy that flows into the segment of the wire of length $L$ (per unit time)

The amount of energy dissipated in this segment in unit time (i.e., Joule heating) is

$$\int_V \vec{j}\cdot\vec{E}\,dV = jE\ \pi a^2 L$$

$$= \frac{I}{\pi a^2}\cdot\frac{I}{\pi a^2 \sigma}\cdot \pi a^2 L$$

$$= \underline{\underline{\frac{I^2 L}{\pi a^2 \sigma}}},$$

which is equal to amount of energy that flows into the segment of length $L$ per unit time.

3 (a) Consider the rate of change of the energy of the electromagnetic field in a given volume $V$:

$$\frac{\partial}{\partial t}\int_V w\,dV = \frac{\partial}{\partial t}\int \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})\,dV$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon} \qquad = \frac{1}{2}\frac{\partial}{\partial t}\int \left(\frac{\vec{D}^2}{\varepsilon} + \frac{\vec{B}^2}{\mu}\right) dV$$

$$\vec{H} = \frac{\vec{B}}{\mu} \qquad = \int \left(\frac{1}{\varepsilon}\vec{D}\cdot\frac{\partial\vec{D}}{\partial t} + \frac{1}{\mu}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right) dV$$

$$= \int \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) dV .$$

Using Maxwell's equations,

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{and} \quad \vec{\nabla}\times\vec{H} = \vec{j} + \frac{\partial\vec{D}}{\partial t},$$

we obtain:

$$\int_V \left[\vec{E}\cdot(\vec{\nabla}\times\vec{H}-\vec{j}) - \vec{H}\cdot(\vec{\nabla}\times\vec{E})\right] dV$$

$$= \int_V \left[\vec{E}\cdot(\vec{\nabla}\times\vec{H}) - \vec{H}\cdot(\vec{\nabla}\times\vec{E})\right] dV - \int_V \vec{j}\cdot\vec{E}\, dV$$

Using the vector identity

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H}),$$

we have:

$$-\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\, dV - \int_V \vec{j}\cdot\vec{E}\, dV$$

and transforming the first term using Gauss's theorem, we find:

$$-\oint_S (\vec{E}\times\vec{H})\cdot d\vec{S} - \int_V \vec{j}\cdot\vec{E}\, dV$$

Introducing the Poynting vector

$$\vec{S} = \vec{E}\times\vec{H},$$

we finally have:

$$\frac{\partial}{\partial t}\int_V w\, dV = -\oint_S \vec{S}\cdot d\vec{S} - \int_V \vec{j}\cdot\vec{E}\, dV.$$

The first term on the right-hand side describes the rate at which the electromagnetic energy in the volume V is lost due to its flow across the surface S that surrounds V. The Poynting vector $\vec{S}$ is the energy flow density. The second term on the right-hand side is the rate at which the energy is dissipated inside V due to the work by the electric field on moving the charges. This is the rate of Joule heating of the medium.

(b) $\vec{D} = \frac{It}{\pi a^2}\vec{k}$ , $\vec{H} = \frac{I\rho}{2\pi a^2}\hat{\varphi}$ .

i. $\vec{\nabla}\cdot\vec{D} = 0$ (no free charges inside the capacitor)

Using the divergence in cylindrical coordinates,

$$\vec{\nabla}\cdot\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\varphi}{\partial\varphi} + \frac{\partial D_z}{\partial z}$$

$$D_\rho = D_\varphi = 0, \qquad D_z = \frac{It}{\pi a^2}$$

$\Rightarrow$ $\vec{\nabla}\cdot\vec{D} = 0$ , as required.

Maxwell's 2nd equation: $\vec{\nabla}\cdot\vec{B} = 0$

For the dielectric we have $\vec{B} = \mu_0\vec{H}$, so this equation is equivalent to $\vec{\nabla}\cdot\vec{H} = 0$.

$$H_\rho = 0, \qquad H_\varphi = \frac{I\rho}{2\pi a^2}, \qquad H_z = 0 .$$

Hence $\vec{\nabla}\cdot\vec{H} = \frac{1}{\rho}\frac{\partial}{\partial\varphi}\left(\frac{I\rho}{2\pi a^2}\right) = 0$, as expected.

Third Maxwell's equation: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ . (*)

$$\vec{E} = \frac{1}{\varepsilon}\vec{D}, \qquad \vec{B} = \mu\vec{H}.$$

The left-hand side :

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\vec{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\varphi & E_z \end{vmatrix} = \begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\vec{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{It}{\pi a^2\varepsilon} \end{vmatrix} = 0$$

(since $E_z$ does not depend on either $\rho$ or $\varphi$).

The right-hand side :

$$-\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t} = -\mu_0\frac{\partial}{\partial t}\left(\frac{I\rho}{2\pi a^2}\hat{\varphi}\right) = 0,$$ hence, (*) holds.

Maxwell's fourths equation: $\vec{\nabla}\times\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$.

$\vec{j} = 0$, since there are no free currents inside the dielectric.

The left-hand side is:

$$\vec{\nabla}\times\vec{H} = \begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\vec{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\varphi & H_z \end{vmatrix} = \begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\vec{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & \frac{I\rho^2}{2\pi a^2} & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\hat{\rho}\left(-\frac{\partial}{\partial z}\frac{I\rho^2}{2\pi a^2}\right) + \frac{1}{\rho}\vec{k}\,\frac{\partial}{\partial\rho}\left(\frac{I\rho^2}{2\pi a^2}\right)$$

$$= 0 + \frac{1}{\rho}\hat{k}\,\frac{2I\rho}{2\pi a^2} = \frac{I}{\pi a^2}\vec{k}.$$

The right-hand side:

$$\frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{It}{\pi a^2}\vec{k}\right) = \frac{I}{\pi a^2}\vec{k}.$$

Hence, we see that $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$, as it should be.

ii. The energy of the electromagnetic field inside the capacitor is

$$\int_V w\,dV = \frac{1}{2}\int_V \left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)dV$$

$\vec{D} = \varepsilon\vec{E}$
$\vec{B} = \mu_0\vec{H}$

$$= \frac{1}{2}\int_V\left(\frac{D^2}{\varepsilon} + \mu_0 H^2\right)dV$$

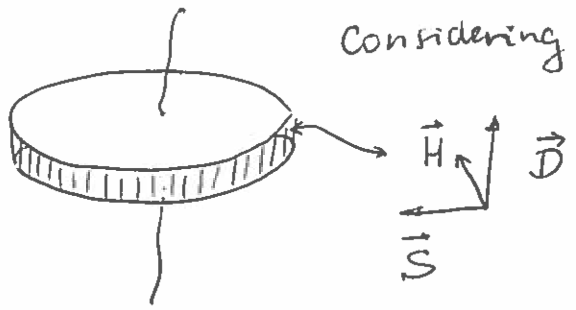
$$= \frac{1}{2}\int_0^a\left(\frac{1}{\varepsilon}\frac{I^2t^2}{\pi^2a^4} + \mu_0\frac{I^2\rho^2}{4\pi^2a^4}\right)\underbrace{2\pi\rho\,d\rho\,d}_{\substack{\text{volume of}\\ \text{cylindrical shell}\\ \text{of radius }\rho\text{, thickness }d\rho\\ \text{and height }d}}$$

$$= \frac{1}{2}\,\frac{1}{\varepsilon}\,\frac{I^2 t^2}{\pi^2 a^4}\,\pi a^2 d + \frac{1}{2}\mu_0 \frac{I^2}{4\pi^2 a^4}\, 2\pi\, \frac{a^4}{4}\, d$$

$$= \underbrace{\frac{I^2 t^2 d}{2\varepsilon\pi a^2}}_{\text{electric energy changes in time.}} + \underbrace{\frac{\mu_0 I^2 d}{16\pi}}_{\text{magnetic energy remains constant}}$$

The rate of change of the electromagnetic energy is

$$\frac{\partial}{\partial t}\int_V w\,dV = \frac{\partial}{\partial t}\left(\frac{I^2 t^2 d}{2\varepsilon\pi a^2}\right) = \frac{I^2 t\, d}{\varepsilon\pi a^2}. \qquad (*)$$




Considering Poynting vector at the edge of the capacitor:

$\vec{D}$ - vertically upwards
$\vec{H}$ - tangential to the circular plates, into the page
$\vec{E}$ parallel to $\vec{D}$

$\vec{S} = \vec{E}\times\vec{H}$ is radial, towards the centre.

$$S = EH = \frac{D}{\varepsilon} H = \frac{It}{\varepsilon\pi a^2}\,\frac{Ia}{2\pi a^2}$$

$$= \frac{I^2 t}{2\varepsilon\pi^2 a^3}$$

Flux of electromagnetic energy across the cylindrical boundary of the capacitor:

$$\oint_S \vec{S}\cdot d\vec{S} = -S\,\underbrace{2\pi a d}_{\text{area of cylindrical surface}} = -\frac{I^2 t}{2\varepsilon\pi^2 a^3}\, 2\pi a d$$

$$= -\frac{I^2 t d}{\varepsilon\pi a^2} \qquad (**)$$

$\Rightarrow \quad \frac{\partial}{\partial t}\int_V w\,dV = -\oint \vec{S}\cdot d\vec{S}$, as expected (from $(*)$ and $(**)$).

④ (a) $\vec{A} = \frac{f'(u)}{vr}\vec{k}$ and $\varphi = \left[\frac{f'(u)}{r} + \frac{v\,f(u)}{r^2}\right]\frac{z}{r}$ ,

where $u = t - r/v$ and $v = \frac{1}{\sqrt{\varepsilon\mu}}$.

To verify that $\vec{A}$ satisfies the equation

$$\nabla^2\vec{A} - \frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0$$

we use the Laplacian in spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \underbrace{\ldots}_{}$$

These terms can be ignored, since $\vec{A}$ depends only on $r$ (and $t$), but not on $\theta$ or $\varphi$.

In $\vec{A}$, $\vec{k}$ is a constant vector, so it is not affected by the derivatives. Denoting $\frac{f'(u)}{vr} = A_z$, we have:

$$\frac{\partial A_z}{\partial r} = \frac{\partial}{\partial r}\frac{f'(u)}{vr} = -\frac{1}{vr^2}f'(u) + \frac{1}{vr}f''(u)\frac{\partial u}{\partial r}$$

$$= -\frac{1}{vr^2}f'(u) + \frac{1}{vr}f''(u)\left(-\frac{1}{v}\right)$$

$$= -\frac{1}{vr^2}f'(u) - \frac{1}{v^2r}f''(u)$$

$$\frac{\partial u}{\partial r} = \frac{\partial(t - r/v)}{\partial r} = -\frac{1}{v}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_z}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{1}{v}f'(u)\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{r}{v^2}f''(u)\right)$$

$$= -\frac{1}{v}\frac{1}{r^2}f''(u)\frac{\partial u}{\partial r} - \frac{1}{v^2}\frac{1}{r^2}\left[f''(u) + r\,f'''(u)\frac{\partial u}{\partial r}\right]$$

$$= \cancel{\frac{1}{v^2}\frac{1}{r^2}f''(u)} - \cancel{\frac{1}{v^2}\frac{1}{r^2}f''(u)} + \frac{1}{v^3}\frac{1}{r}f'''(u)$$

$$= \frac{1}{v^3}\frac{f'''(u)}{r}.$$

The time derivatives are:

$$\frac{\partial A_z}{\partial t} = \frac{\partial}{\partial t}\,\frac{f'(u)}{vr} = \frac{1}{vr}\,f''(u)\,\frac{\partial u}{\partial t} \qquad \left|\; \frac{\partial u}{\partial t} = \frac{\partial(t - r/v)}{\partial t} = 1 \right.$$

$$= \frac{1}{vr}\,f''(u)$$

$$\frac{\partial^2 A_z}{\partial t^2} = \frac{1}{vr}\,f'''(u)\,\frac{\partial u}{\partial t} = \frac{1}{vr}\,f'''(u).$$

Hence,

$$\nabla^2 A_z - \frac{1}{v^2}\frac{\partial^2 A_z}{\partial t^2} = \frac{1}{v^3}\,\frac{f'''(u)}{r} - \frac{1}{v^2}\,\frac{f'''(u)}{vr} = 0,$$

<u>as required.</u>

In the expression for the scalar potential,

$$\frac{z}{r} = \cos\theta \quad\Rightarrow\quad \varphi = \left[\frac{f'(u)}{r} + \frac{v\,f(u)}{r^2}\right]\cos\theta$$

In the Laplacian we now need to keep terms with derivatives with respect to $r$ and $\theta$:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \dots$$

The radial part acts only on the factor in $\varphi$ in $[\dots]$.

$$\frac{\partial}{\partial r}\left[\frac{f'(u)}{r} + \frac{v\,f(u)}{r^2}\right] = -\frac{1}{r^2}f'(u) + \frac{1}{r}f''(u)\left(-\frac{1}{v}\right)$$

$\left(\frac{\partial u}{\partial r}\right)$

$$-\frac{2v}{r^3}f(u) + \frac{v}{r^2}f'(u)\left(-\frac{1}{v}\right)$$

$$= -\frac{2}{r^2}f'(u) - \frac{1}{vr}f''(u) - \frac{2v}{r^3}f(u)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\,r^2\left[-\frac{2}{r^2}f'(u) - \frac{1}{vr}f''(u) - \frac{2v}{r^3}f(u)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[-2f'(u) - \frac{r}{v}f''(u) - \frac{2v}{r}f(u)\right]$$

$$= -\frac{1}{r^2}\Big[ 2f''(u)\left(-\frac{1}{v}\right) + \frac{1}{v}f''(u) + \frac{r}{v}f'''(u)\left(-\frac{1}{v}\right) - \frac{2v}{r^2}f(u)$$

$$+ \frac{2v}{r}f'(u)\left(-\frac{1}{v}\right)\Big]$$

$$= -\frac{1}{r^2}\left[ -\frac{f''(u)}{v} - \frac{r}{v^2}f'''(u) - \frac{2v}{r^2}f(u) - \frac{2f'(u)}{r}\right]$$

$$= \frac{1}{r^2}\left[\frac{2v}{r^2}f(u) + \frac{2f'(u)}{r} + \frac{f''(u)}{v} + \frac{rf'''(u)}{v^2}\right]$$

The derivatives in the angular part:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\cos\theta\right) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta(-\sin\theta)\right)$$

$$= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta) = -\frac{2}{\sin\theta}\sin\theta\cos\theta = -2\cos\theta$$

Therefore,

$$\nabla^2\varphi = \frac{1}{r^2}\left[\frac{2v}{r^2}f(u) + \frac{2f'(u)}{r} + \frac{f''(u)}{v} + \frac{rf'''(u)}{v^2}\right]\cos\theta$$

$$- \frac{2}{r^2}\left[\frac{f'(u)}{r} + \frac{vf(u)}{r^2}\right]\cos\theta$$

$$= \frac{1}{r^2}\left[\frac{f''(u)}{v} + \frac{rf'''(u)}{v^2}\right]\cos\theta$$

Terms containing $f(u)$ and $f'(u)$ cancel.

The time derivatives:

$$\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial}{\partial t}\frac{\partial}{\partial t}\left[\frac{f'(u)}{r} + \frac{vf(u)}{r^2}\right]\cos\theta$$

$$= \left[\frac{f'''(u)}{r} + \frac{vf''(u)}{r^2}\right]\cos\theta$$

since $\frac{\partial u}{\partial t} = 1$

Hence,

$$\nabla^2\varphi - \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{1}{r^2}\left[\cancel{\frac{f''(u)}{v}} + \cancel{\frac{rf'''(u)}{v^2}}\right]\cos\theta - \frac{1}{v^2}\left[\cancel{\frac{vf''(u)}{r^2}} + \cancel{\frac{f'''(u)}{r}}\right]$$

$$\times\cos\theta = 0.$$

Hence, we see that both $\vec{A}$ and $\varphi$ satisfy the wave equation. 15

(b) The Lorenz condition is

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{v^2}\frac{\partial \varphi}{\partial t} = 0 .$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_z}{\partial z} = \frac{\partial A_z}{\partial r}\frac{\partial r}{\partial z}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial z} = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \frac{z}{r} = \cos\theta .$$

$$= \frac{\partial}{\partial r}\left(\frac{f'(u)}{vr}\right)\cdot \cos\theta$$

$$= \left[-\frac{f'(u)}{vr^2} + \frac{f''(u)}{vr}\left(-\frac{1}{v}\right)\right]\cos\theta$$

$$= -\left[\frac{f'(u)}{vr^2} + \frac{f''(u)}{v^2 r}\right]\cos\theta .$$

$$\frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial t}\left[\frac{f'(u)}{r} + \frac{v f(u)}{r^2}\right]\cos\theta$$

$$= \left[\frac{f''(u)}{r} + \frac{v f'(u)}{r^2}\right]\cos\theta$$

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{v^2}\frac{\partial\varphi}{\partial t} = -\left[\cancel{\frac{f'(u)}{vr^2}} + \cancel{\frac{f''(u)}{v^2 r}}\right]\cos\theta + \frac{1}{v^2}\left[\cancel{\frac{f''(u)}{r}} + \cancel{\frac{v f'(u)}{r^2}}\right]$$

$$\times \cos\theta = 0,$$

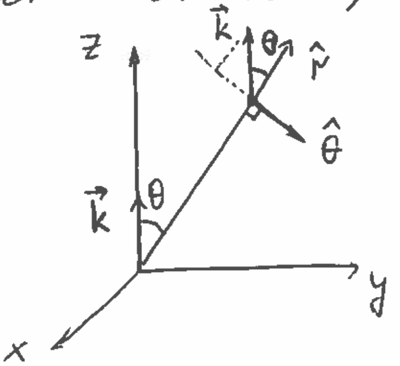
as required.

(c) To find $\vec{B}$ from $\vec{B} = \vec{\nabla}\times\vec{A}$, let us express $\vec{k}$ in terms of the unit vectors of the spherical coordinates system:

Component of $\vec{k}$ along $\hat{r}$: $\cos\theta$

Component of $\vec{k}$ along $\hat{\theta}$: $-\sin\theta$

$\Rightarrow \quad \vec{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ .



Hence, $\quad \vec{A} = \frac{f'(u)}{vr}\vec{k} = \underbrace{\frac{f'(u)}{vr}\cos\theta}_{A_r}\,\hat{r} - \underbrace{\frac{f'(u)}{vr}\sin\theta}_{A_\theta}\,\hat{\theta}$

$$\vec{B} = \begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & \underbrace{r\sin\theta A_\varphi}_{=0} \end{vmatrix} = \frac{1}{r^2\sin\theta}\hat{r}\Big(\underbrace{-\frac{\partial(rA_\theta)}{\partial\varphi}}_{=0}\Big)$$

$$- \frac{1}{r\sin\theta}\hat{\theta}\Big[\underbrace{-\frac{\partial A_r}{\partial\varphi}}_{=0}\Big] + \frac{1}{r}\hat{\varphi}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{1}{r}\hat{\varphi}\left[-\frac{\partial}{\partial r}\left(\frac{f'(u)}{v}\right)\sin\theta - \frac{f'(u)}{vr}\frac{\partial\cos\theta}{\partial\theta}\right]$$

$$= \frac{1}{r}\hat{\varphi}\left[-\frac{f''(u)}{v}\left(-\frac{1}{v}\right)\sin\theta + \frac{f'(u)}{vr}\sin\theta\right]$$

$$\Rightarrow \qquad \vec{B} = \frac{1}{r}\left[\frac{f''(u)}{v^2} + \frac{f'(u)}{vr}\right]\sin\theta\,\hat{\varphi}\,.$$

The electric field $\quad \vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}$

Using the expression for the gradient in spherical polar coordinates, we have:

$$\vec{E} = -\frac{\partial}{\partial r}\left[\frac{f'(u)}{r} + \frac{v\,f(u)}{r^2}\right]\cos\theta\,\hat{r} - \frac{1}{r}\left[\frac{f'(u)}{r} + \frac{v\,f(u)}{r^2}\right]\frac{\partial\cos\theta}{\partial\theta}\,\hat{\theta}$$

$$-\frac{\partial}{\partial t}\left[\frac{f'(u)}{vr}\cos\theta\,\hat{r} - \frac{f'(u)}{vr}\sin\theta\,\hat{\theta}\right]$$

$$= \left[\frac{1}{r^2}f'(u) - \frac{1}{r}\cancel{f''(u)\left(-\frac{1}{v}\right)} + \frac{2v\,f(u)}{r^3} - \frac{v\,f'(u)}{r^2}\left(-\frac{1}{v}\right) - \cancel{\frac{f''(u)}{vr}}\right]\cos\theta\,\hat{r} + \left[\frac{f'(u)}{r^2} + \frac{v\,f(u)}{r^3} + \frac{f''(u)}{vr}\right]\sin\theta\,\hat{\theta}$$

$$\Rightarrow \quad \vec{E} = \left[ \frac{2f'(u)}{r^2} + \frac{2v\,f(u)}{r^3} \right] \cos\theta \, \hat{r}$$

$$+ \left[ \frac{f''(u)}{vr} + \frac{f'(u)}{r^2} + \frac{v\,f(u)}{r^3} \right] \sin\theta \, \hat{\theta} \, .$$

Note that at large $r$ both $\bar{E}$ and $\bar{B}$ decrease as $\frac{1}{r}$ and they are given by

$$\vec{E} \simeq \frac{f''(u)}{vr} \sin\theta \, \hat{\theta} \, ,$$

$$\bar{B} \simeq \frac{f''(u)}{v^2 r} \sin\theta \, \hat{\varphi} \, .$$

They obey the relation

$$\vec{B} = \frac{1}{v} \, \hat{r} \times \vec{E}$$

and describe the electromagnetic wave radiated by a source at the origin. This source is in fact a time-varying electric dipole (known as the Herzian dipole), with the function $f(t)$ related to its magnitude. Indeed, considering $\varphi$ at small $r$ we see that

$$\varphi \simeq \frac{v\,f(u)}{r^2} \cos\theta \simeq \frac{v\,f(t)}{r^2} \cos\theta \qquad (*)$$

[ $u = t - r/v$ , so for small $r$ , $u \simeq t$ ].
Electrostatic potential of the dipole is

$$\varphi = \frac{1}{4\pi\varepsilon_0} \frac{\bar{p} \cdot \bar{r}}{r^3} \, ,$$

so we see that in $(*)$, $\vec{p} = 4\pi\varepsilon_0 \, v \, f(t) \, \vec{k}$ (time-changing dipole moment along the $z$ axis).