Electromagnetic Theory AMA3001 — Problem sheet 9

Electromagnetic waves.

Homework

1. (a) Starting from Maxwell's equations for a linear medium, show how $w = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H})$, $\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H}$ and $\boldsymbol{j}\cdot\boldsymbol{E}$ can be interpreted in terms of energy balance within a volume.

   (b) Show that Maxwell's equations for a non-conducting, uniform, charge-free medium possess plane-wave solutions of the form

   $$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0 \exp[i(k\boldsymbol{n}\cdot\boldsymbol{r} - \omega t)],$$
   $$\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_0 \exp[i(k\boldsymbol{n}\cdot\boldsymbol{r} - \omega t)],$$

   where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are constant complex vectors, ω and k are real constants, and $\boldsymbol{n}$ is a real unit vector.

   Prove that:

   i. $k = \omega\sqrt{\varepsilon\mu}$;

   ii. $\boldsymbol{n}\cdot\boldsymbol{E}_0 = \boldsymbol{n}\cdot\boldsymbol{H}_0 = 0$;

   iii. $\boldsymbol{H}_0 = \dfrac{k}{\omega\mu}\boldsymbol{n}\times\boldsymbol{E}_0$.

   (c) Evaluate w and $\boldsymbol{S}$ for the plane-wave solutions and interpret the answer in terms of energy flow.

   [Exam 2004 Question 7]

2. By writing

   $$\boldsymbol{E}_0 = E_\rho\boldsymbol{\rho} + E_s\boldsymbol{s},$$

   where $(\boldsymbol{\rho},\boldsymbol{s},\boldsymbol{n})$ form a right-handed set of real, orthogonal, unit vectors, in Question 1 above, explain what is meant by

   (a) linear polarization;

   (b) circular polarization, both left-handed and right-handed.

   [Exam 2005 Question 6 part (b); part (a) was the same as part (b) of Question 1 above.]

3. A plane electromagnetic wave is incident upon the plane boundary between two homogeneous, isotropic, transparent, semi-infinite, nonmagnetic (i.e., $\mu = \mu_0$) media. Obtain expressions for the components (perpendicular and parallel to the plane of incidence) of the electric vectors of the reflected and transmitted waves. From these expressions show how it is possible to obtain a reflected beam of plane polarized light from an unpolarized incident beam.

   The following may be assumed :

   (a) the law of reflection and Snell's law of refraction;

   (b) the relation $\boldsymbol{H} = \boldsymbol{k}\times\boldsymbol{E}/(\omega\mu_0)$ between the magnetic and electric vectors of a plane electromagnetic wave in such a medium, where $\boldsymbol{k}$ is the wave vector;

   (c) the trigonometric relation

   $$\frac{\sin A\cos A - \sin B\cos B}{\sin A\cos A + \sin B\cos B} = \frac{\tan(A-B)}{\tan(A+B)}.$$

   [Exam 2003 Question 5]

4. A rectangular waveguide consists of perfectly conducting walls at $x = 0$, $x = a$, $y = 0$ and $y = b$. The space inside the waveguide is vacuum.

   An electromagnetic wave propagates down the waveguide in the z direction. Its electric and magnetic fields are given by the real parts of

$$\boldsymbol{E}(x, y) \exp[i(k_g z - \omega t)] \quad \text{and} \quad \boldsymbol{H}(x, y) \exp[i(k_g z - \omega t)],$$

   respectively, and $H_z = 0$, so that the wave is transverse magnetic (TM).

   (a) Starting from Maxwell's equations show that all components of the fields can be expressed in terms of E_z.

   (b) Find the differential equation satisfied by E_z.

   (c) Verify that

$$E_z(x, y) = A \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right),$$

   where A is an arbitrary constant and m, n are integers, satisfies the appropriate boundary conditions and also satisfies the differential equation of part (b) provided

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \frac{\omega^2}{c^2} - k_g^2,$$

   where c is the speed of light in vacuum.

   [Exam 2005 Question 7]

5. (a) State the boundary conditions on the fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ at the interface between two media.

   (b) A cylindrical waveguide consists of a conductor with the space inside being vacuum. An electromagnetic wave given by the real parts of

$$\boldsymbol{E}(\rho, \psi) \exp[i(kz - \omega t)] \quad \text{and} \quad \boldsymbol{H}(\rho, \psi) \exp[i(kz - \omega t)]$$

   propagates down the waveguide in the z direction. The z-axis is the axis of the waveguide and (ρ, ψ, z) are cylindrical polar coordinates.

   i. If $H_z = 0$ show that all the components of $\boldsymbol{E}$ and $\boldsymbol{H}$ are determined once E_z is known.

   ii. Find the differential equation satisfied by E_z.

   [Exam 2003 Question 6]

6. A waveguide is a hollow cylindrical conductor filled by a medium of uniform permittivity ε and permeability μ. Taking the z-axis parallel to the axis of the waveguide, the electric and magnetic fields are expressed as

$$\boldsymbol{\mathcal{E}}(\boldsymbol{r}, t) = \boldsymbol{E} \exp[i(kz - \omega t)] \quad \text{and} \quad \boldsymbol{\mathcal{H}}(\boldsymbol{r}, t) = \boldsymbol{H} \exp[i(kz - \omega t)],$$

   respectively, where $\boldsymbol{E}$ and $\boldsymbol{H}$ do not depend on z.

   (a) Use Maxwell's equations to show that, when $H_z = 0$ (the TM wave),

   i. $\boldsymbol{H}_t = \dfrac{\varepsilon\omega}{k} \boldsymbol{e}_z \times \boldsymbol{E}_t\,;$

   ii. $-\dfrac{iK^2}{k} \boldsymbol{E}_t = \nabla_t E_z\,;$

   iii. $(\nabla_t^2 + K^2) E_z = 0\,,$

where $K^2 = -k^2 + \omega^2 \varepsilon\mu$ and subscript t denotes the transverse vector (in the plane with normal $\boldsymbol{e}_z$). [Hint: use $\boldsymbol{E} = \boldsymbol{E}_t + \boldsymbol{e}_z E_z$, $\nabla = \nabla_t + \boldsymbol{e}_z \partial/\partial z$, $\boldsymbol{H} = \boldsymbol{H}_t$, where $\boldsymbol{e}_z$ is the unit vector along z.]

(b) For a circular waveguide ($0 \le \rho \le a$) the solution for E_z is

$$E_z(\rho, \psi) = \alpha J_n(K\rho)\cos(n\psi),$$

where α is a constant, $J_n(x)$ is a Bessel function, and $n = 0,\ 1,\ 2, \ldots$.
Use the boundary condition satisfied by E_z to obtain the expression

$$k^2 = \varepsilon\mu(\omega^2 - \omega_{nm}^2),$$

and a formula for the cut-off angular frequency ω_{nm}. Explain what is meant by modes in the wave propagation.

[Exam 2006 Question 7]

Further Examples

1. Given the electromagnetic wave

$$\boldsymbol{E} = \boldsymbol{i}E_0 \cos[\omega(\sqrt{\varepsilon\mu}z - t)] + \boldsymbol{j}E_0 \sin[\omega(\sqrt{\varepsilon\mu}z - t)],$$

where E_0 is a constant, find the corresponding magnetic field $\boldsymbol{B}$ and the Poynting vector.

2. Given $E_x = E_y = 0$, $E_z = a\cos kx \cos\omega t$ and $\boldsymbol{H} = 0$ when $t = 0$, in vacuum, i.e., $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\rho = 0$ and $\sigma = 0$, show that

$$H_x = H_z = 0, \quad H_y = -a\sqrt{\varepsilon_0/\mu_0}\sin kx \sin\omega t.$$

Verify that there is no mean flux of energy in this problem, which corresponds to standing or stationary waves.

3. Two monochromatic plane polarized waves of the same frequency propagate along Oz. One is polarized in the x direction with amplitude E_x, the other along the y direction with amplitude E_y. The phase of the second leads that of the first by ψ. Determine the polarization of the resultant wave.

4. Show that when $\sigma/\varepsilon\omega \gg 1$, the electric field in the conducting half-space $z \ge 0$ may be expressed as

$$\boldsymbol{E}(z,t) = \boldsymbol{E}_0 \exp\left(-\sqrt{\frac{\sigma\omega\mu}{2}}z\right)\cos\left(\sqrt{\frac{\sigma\omega\mu}{2}}z - \omega t\right),$$

where $\boldsymbol{E}_0$ has zero z component. Deduce that if $\boldsymbol{E}$ has only a component E_x then the only nonvanishing component of $\boldsymbol{B}$ is B_y, and that B_y and E_x have a phase difference of $\pi/4$.

5. Find the surface charge density and the current per unit width on the surface of a perfect conductor on which plane electromagnetic waves are incident, when the electric vector is (1) perpendicular to the plane of incidence, and (2) parallel to the plane of incidence.

<u>Answers:</u>

4(a). $E_x = \dfrac{k_g}{i(k_g^2 - \varepsilon_0\mu_0\omega^2)}\dfrac{\partial E_z}{\partial x}$, $E_y = \dfrac{k_g}{i(k_g^2 - \varepsilon_0\mu_0\omega^2)}\dfrac{\partial E_z}{\partial y}$, $H_x = -\dfrac{\varepsilon_0\omega}{k_g}E_y = \ldots$, $H_y = \dfrac{\varepsilon_0\omega}{k_g}E_x = \ldots$.

4(b). $\dfrac{\partial^2 E_z}{\partial x^2} + \dfrac{\partial^2 E_z}{\partial y^2} + \left(\dfrac{\omega^2}{c^2} - k_g^2\right)E_z = 0$. 5(b)ii. $\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial E_z}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 E_z}{\partial\psi^2} + \left(\dfrac{\omega^2}{c^2} - k^2\right)E_z = 0$.