AMA3001 Electromagnetic Theory.

Problem sheet 9

SOLUTIONS

① (a) Consider the electromagnetic energy in a fixed volume V (assuming $w = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$ is the energy density):

$$\int_V w\, dV = \frac{1}{2}\int_V (\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})\, dV .$$

It's rate of change,

$$\frac{\partial}{\partial t}\int_V w\, dV = \frac{1}{2}\frac{\partial}{\partial t}\int_V (\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})\, dV$$

can be worked out using Maxwell's equations

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and} \quad \vec{\nabla}\times\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

and the relations $\vec{E} = \dfrac{\vec{D}}{\varepsilon}$, $\vec{H} = \dfrac{\vec{B}}{\mu}$:

$$\frac{\partial}{\partial t}\int_V w\, dV = \frac{1}{2}\frac{\partial}{\partial t}\int_V \left(\frac{\vec{D}\cdot\vec{D}}{\varepsilon} + \frac{\vec{H}\cdot\vec{H}}{\mu}\right) dV$$

$$= \int_V \left(\frac{1}{\varepsilon}\vec{D}\cdot\frac{\partial \vec{D}}{\partial t} + \frac{1}{\mu}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t}\right) dV$$

$$= \int_V \left[\vec{E}\cdot(\vec{\nabla}\times\vec{H} - \vec{j}) - \vec{H}\cdot(\vec{\nabla}\times\vec{E})\right] dV$$

Using the relation

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H}),$$

we find:

$$\frac{\partial}{\partial t}\int_V w\, dV = -\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\, dV - \int_V \vec{j}\cdot\vec{E}\, dV .$$

Applying Gauss's theorem to the 1st term on the right-hand side transforms it into a surface integral over the surface S that surrounds V:

$$\frac{\partial}{\partial t}\int_V w\,dV = -\oint_S \vec{S}\cdot d\vec{s} - \int_V \vec{j}\cdot\vec{E}\,dV , \quad (*)$$

where $\vec{S} = \vec{E}\times\vec{H}$ is the Poynting vector.

The first term on the right-hand side of (*) describes the flow of electromagnetic energy across the surface S from the volume V, the Poynting vector giving the energy flux density. The second term describes the amount of work done by the electromagnetic field on moving the charges, per unit time. Usually (e.g., when Ohm's law $\vec{j} = \sigma\vec{E}$ holds) this term describes the energy loss due to heating, called Joule heating.

(b)
$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i(k\vec{n}\cdot\vec{r}-\omega t)},$$
$$\vec{H}(\vec{r},t) = \vec{H}_0\, e^{i(k\vec{n}\cdot\vec{r}-\omega t)}$$

Maxwell's equations for a nonconducting, charge-free, uniform medium are

$$\vec{\nabla}\cdot\vec{D} = 0 \iff \vec{\nabla}\cdot\vec{E} = 0 \quad (1)\ (\vec{D} = \varepsilon\vec{E})$$

$$\vec{\nabla}\cdot\vec{B} = 0 \iff \vec{\nabla}\cdot\vec{H} = 0 \quad (2)\ (\vec{B} = \mu\vec{H})$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \iff \vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \quad (3)$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \iff \vec{\nabla}\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t}. \quad (4)$$

From (1),

$$\vec{\nabla}\cdot\left(\vec{E}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)}\right) = \vec{E}_0\cdot\vec{\nabla}\left(e^{i(k\vec{n}\cdot\vec{r}-\omega t)}\right)$$

$$= \vec{E}_0\cdot\vec{\nabla}\left(i(k\vec{n}\cdot\vec{r}-\omega t)\right) e^{i(k\vec{n}\cdot\vec{r}-\omega t)} \quad \Big\{ \ \vec{\nabla}(\vec{n}\cdot\vec{r}) = \vec{n}$$

$$= ik\vec{n}\cdot\vec{E}_0\, e^{i(k\vec{n}\cdot\vec{r}-\omega t)} = 0$$

$$\Rightarrow \quad \underline{\underline{\vec{n}\cdot\vec{E}_0 = 0}}.$$ [Electric field is perpendicular to the direction of propagation of the wave]

From (2), similarly,

$$\vec{\nabla}\cdot\vec{H}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)} = ik\vec{n}\cdot\vec{H}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)} = 0$$

$$\Rightarrow \quad \underline{\underline{\vec{n}\cdot\vec{H}_0 = 0}}$$ [Magnetic field is perpendicular to the direction of propagation]

In order to apply (3), let us work out the left-hand side and the right-hand side separately first:

$$\vec{\nabla}\times\vec{E}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)} = \vec{\nabla}\left(e^{i(k\vec{n}\cdot\vec{r}-\omega t)}\right)\times\vec{E}_0$$

$$= ik\vec{n}\times\vec{E}_0\, e^{i(k\vec{n}\cdot\vec{r}-\omega t)}$$

$$-\mu\frac{\partial}{\partial t}\left(\vec{H}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)}\right) = +i\mu\omega\vec{H}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)}$$

{ $\vec{E}_0$ = const, terms in the cross product should not be swapped

Hence,

$$ik\vec{n}\times\vec{E}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)} = i\mu\omega\vec{H}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)}$$

$$\Rightarrow \quad \vec{H}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)} = \frac{k}{\omega\mu}\vec{n}\times\vec{E}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)} \qquad (5)$$

$$\Rightarrow \quad \underline{\underline{\vec{H}_0 = \frac{k}{\omega\mu}\vec{n}\times\vec{E}_0}}.$$ [Magnetic field is perpendicular to the electrical field. -

Finally, using (4) and proceeding similarly, we find:

$$ik\vec{n}\times\vec{H}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)} = -i\varepsilon\omega\vec{E}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)}$$

$$\Rightarrow \quad \vec{E}_0 = -\frac{k}{\omega\varepsilon}\,\vec{n}\times\vec{H}_0$$

Substituting this into the expression for $\vec{H}_0$, we have:

$$\vec{H}_0 = \frac{k}{\omega\mu}\left(-\frac{k}{\omega\varepsilon}\right)\vec{n}\times(\vec{n}\times\vec{H}_0)$$

$$\vec{H}_0 = -\frac{k^2}{\omega^2\varepsilon\mu}\left[\vec{n}\,(\underbrace{\vec{n}\cdot\vec{H}_0}_{=0}) - \vec{H}_0(\underbrace{\vec{n}\cdot\vec{n}}_{=1})\right]$$

$$\Rightarrow \quad \vec{H}_0 = \frac{k^2}{\omega^2\varepsilon\mu}\vec{H}_0,$$

$$\Rightarrow \quad \frac{k^2}{\omega^2\varepsilon\mu} = 1 \quad \Leftrightarrow \quad k^2 = \omega^2\varepsilon\mu$$

$$\Rightarrow \quad \underline{\underline{k = \omega\sqrt{\varepsilon\mu}}}.$$

[Alternatively, one could take the time derivative of (4) to obtain:

$$\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}) = \varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\vec{\nabla}\times\frac{\partial\vec{H}}{\partial t} = \varepsilon\frac{\partial^2\vec{E}}{\partial t^2},$$ and

using (3): $$\frac{1}{\mu}\vec{\nabla}\times(-\vec{\nabla}\times\vec{E}) - \varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0}) - \nabla^2\vec{E}$$

$$\Rightarrow \quad \nabla^2\vec{E} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = 0$$: wave equation.

Substituting $\vec{E} = \vec{E}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)}$ yields (after some algebra): $k^2 - \varepsilon\mu\omega^2 = 0 \Rightarrow k = \omega\sqrt{\varepsilon\mu}$.]

(c) Taking the fields as real in this section, ☺

$$\vec{E} = Re\left[\vec{E}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)}\right]$$

$$\vec{H} = Re\left[\vec{H}_0 e^{i(k\vec{n}\cdot\vec{r}-\omega t)}\right],$$

we find the energy density

$$w = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$$

$$= \frac{1}{2}(\varepsilon\vec{E}\cdot\vec{E} + \mu\vec{H}\cdot\vec{H}).$$

From equation (5) (bottom of page 3), we have:

$$\vec{H} = \frac{k}{\omega\mu}\vec{n}\times\vec{E},$$

which means that

$$\vec{H}^2 = \frac{k^2}{\omega^2\mu^2}\vec{E}^2 = \frac{\varepsilon\mu}{\mu^2}\vec{E} = \frac{\varepsilon}{\mu}\vec{E}^2,$$

so we have:

$$w = \frac{1}{2}\left(\varepsilon\vec{E}^2 + \mu\frac{\varepsilon}{\mu}\vec{E}^2\right),$$

$$\underline{w = \varepsilon\vec{E}^2}.$$

For the Poynting vector, we have

$$\vec{S} = \vec{E}\times\vec{H} = \frac{k}{\omega\mu}\vec{E}\times(\vec{n}\times\vec{E})$$

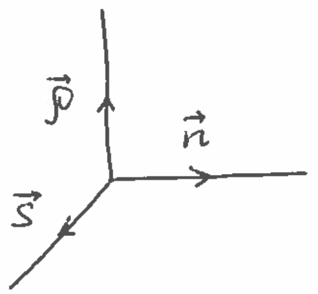
$$= \frac{k}{\omega\mu}\left[\vec{n}(\vec{E}\cdot\vec{E}) - \vec{E}(\underbrace{\vec{E}\cdot\vec{n}}_{=0})\right]$$

$$\Rightarrow \underline{\vec{S} = \frac{k}{\omega\mu}\vec{E}^2\vec{n}}.$$ (energy propagates in the direction of $\vec{n}$)

$$\Rightarrow |\vec{S}| = \frac{k}{\omega\mu}\vec{E}^2 = \frac{k}{\omega\varepsilon\mu}\varepsilon\vec{E}^2 = \frac{\sqrt{\varepsilon\mu}}{\varepsilon\mu}\varepsilon\vec{E}^2 = \frac{1}{\sqrt{\varepsilon\mu}}\varepsilon\vec{E}^2.$$

Therefore $\underline{\underline{|\vec{S}| = cw}}$, where $c = \frac{1}{\sqrt{\varepsilon\mu}}$ is the speed of the wave, which is the amount of energy that flows across unit area in unit time.

② (a)



$(\vec{p}, \vec{s}, \vec{n})$ is the right-handed set of vectors (rotation from $\vec{p}$ to $\vec{s}$ is anticlockwise, if viewed from top of $\vec{n}$).

$\vec{n}$ is the direction of propagation of the wave.

The electric field with the amplitude

$$\vec{E}_0 = E_p \vec{p} + E_s \vec{s}$$

is in the $(\vec{p}, \vec{s})$ plane, perpendicular to $\vec{n}$.

In the case of <u>linear</u> polarisation, the $\vec{p}$ and $\vec{s}$ components of the electric field vary <u>in phase</u>. This is true if the <u>phases</u> of the complex amplitudes E_p and E_s are the same, i.e.,

$$E_p = |E_p| e^{-i\alpha}$$
$$E_s = |E_s| e^{-i\alpha}$$

In this case

$$\vec{E} = \vec{E}_0 e^{i(k\vec{n}\cdot\vec{r} - \omega t)}$$
$$= |E_p| \vec{p} e^{i(k\vec{n}\cdot\vec{r} - \omega t - \alpha)} + |E_s| \vec{s} e^{i(k\vec{n}\cdot\vec{r} - \omega t - \alpha)},$$

and the true (real) electric field oscillates as

$$\vec{E} = Re\left[\vec{E}_0 e^{i(k\vec{n}\cdot\vec{r} - \omega t)}\right]$$
$$= \underbrace{(|E_p| \vec{p} + |E_s| \vec{s})} \cos(k\vec{n}\cdot\vec{r} - \omega t - \alpha).$$

This vector determines the direction of polarisation in the $(\vec{p}, \vec{s})$ plane.

(b) The light (or electromagnetic wave) is circularly polarised when its two mutually perpendicular components are $\frac{\pi}{2}$ out of phase, but have equal amplitudes.

This occurs if
$$E_p = |E_p| e^{-i\alpha}$$
$$E_s = |E_s| e^{-i\alpha \pm i\pi/2}, \quad |E_s| = |E_p|$$

The real field then is
$$\vec{E} = Re\left[\vec{E}_0 e^{i(k\vec{n}\cdot\vec{r} - \omega t)}\right]$$
$$= Re\left[\, |E_p| \vec{p}\, e^{i(k\vec{n}\cdot\vec{r} - \omega t - \alpha)} + |E_p| \vec{s}\, e^{i(k\vec{n}\cdot\vec{r} - \omega t - \alpha \pm \pi/2)}\right]$$

$$= |E_p| \vec{p} \cos(k\vec{n}\cdot\vec{r} - \omega t - \alpha) + |E_p| \vec{s} \cos(k\vec{n}\cdot\vec{r} - \omega t - \alpha \pm \pi/2)$$

$$\left|\begin{array}{l} \cos(\theta + \frac{\pi}{2}) \\ = -\sin\theta \\ \cos(\theta - \frac{\pi}{2}) = \sin\theta \end{array}\right.$$

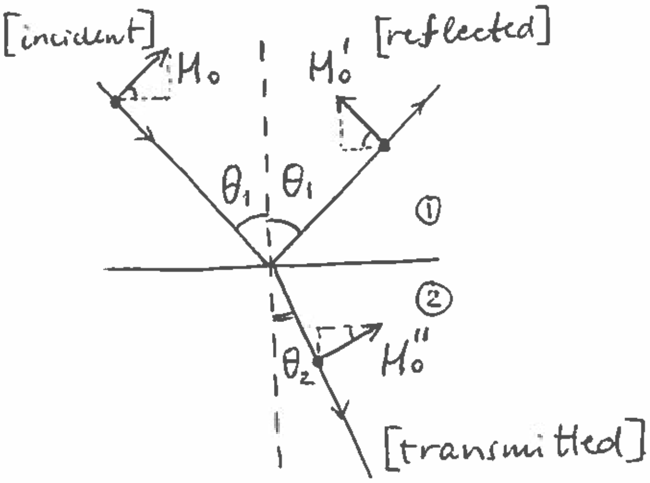
$$= |E_p| \vec{p} \cos(k\vec{n}\cdot\vec{r} - \omega t - \alpha) \mp |E_p| \vec{s} \sin(k\vec{n}\cdot\vec{r} - \omega t - \alpha)$$

$$\Rightarrow \begin{cases} E_p(t) = |E_p| \cos(\omega t + \alpha - k\vec{n}\cdot\vec{r}) \\ E_s(t) = \pm |E_p| \sin(\omega t + \alpha - k\vec{n}\cdot\vec{r}). \end{cases}$$

For the upper sign, the electric field vector describes a circle in the $(\vec{p}, \vec{s})$ plane, rotating anticlockwise (when viewed from the "tip of $\vec{n}$", i.e., with the wave propagating towards the observer). This is a <u>right circularly polarised</u> wave.

For the lower, "−" sign, the field rotates clockwise, and the wave is <u>left circularly polarised</u>.

③ Let us first consider a wave polarised in the direction perpendicular to the plane of incidence (which is taken to be the page plane).



$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$ - Snell's law

Let the electric field amplitudes of the incident, reflected and transmitted waves (E_0, E_0' and E_0'') be into the page. The magnetic fields amplitudes H_0, H_0' and H_0'' are then as shown.

From the boundary condition

$$E_{1t} = E_{2t}$$

(index t denotes tangential components), we have:

$$E_0 + E_0' = E_0'' \quad . \qquad (1)$$

From the boundary condition

$$H_{1t} = H_{2t} \quad .$$

we have:

$$H_0 \cos\theta_1 - H_0' \cos\theta_1 = H_0'' \cos\theta_2 \quad . \qquad (2)$$

From

$$\vec{H} = \frac{1}{\omega\mu_0} \vec{k} \times \vec{E}$$

the magnitudes of $\vec{H}$ and $\vec{E}$ are related by

$$H = \frac{k}{\omega\mu_0} E = \frac{1}{c\mu_0} E = \frac{\sqrt{\varepsilon\mu_0}}{\mu_0} E = \sqrt{\frac{\varepsilon}{\mu_0}}\, E \, ,$$

where ε is the permittivity of the medium.
Hence, equation (2) gives:

$$\sqrt{\varepsilon_1}\,(E_0 - E_0')\cos\theta_1 = \sqrt{\varepsilon_2}\cos\theta_2\, E_0'' \, , \qquad (2')$$

where ε_1 and ε_2 are the permittivities of media 1 and 2.

Substituting E_0'' from (1) into (2') and using Snell's law ($\sqrt{\varepsilon_2}/\sqrt{\varepsilon_1} = \dfrac{\sin\theta_1}{\sin\theta_2}$), we have:

$$(E_0 - E_0')\cos\theta_1 = \frac{\sin\theta_1}{\sin\theta_2}\cos\theta_2\,(E_0 + E_0')$$

$$E_0\cos\theta_1 - E_0\frac{\sin\theta_1}{\sin\theta_2}\cos\theta_2 = E_0'\cos\theta_1 + \frac{\sin\theta_1}{\sin\theta_2}\cos\theta_2\,E_0'$$

$$E_0(\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2) = E_0'(\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2)$$

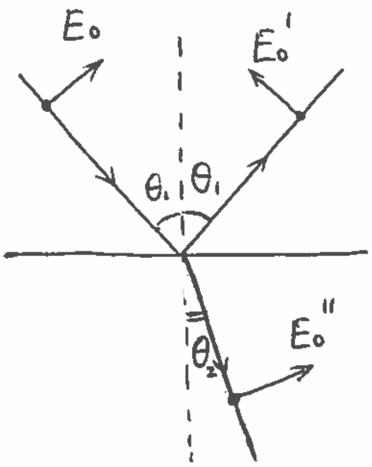
$$\Rightarrow \quad \underline{\underline{E_0' = \frac{\sin(\theta_2-\theta_1)}{\sin(\theta_2+\theta_1)}E_0}}\,,$$ for the amplitude of the reflected wave.

Substituting this into (1), we find:

$$E_0'' = E_0 + \frac{\sin(\theta_2-\theta_1)}{\sin(\theta_2+\theta_1)}E_0$$

$$= \frac{\sin(\theta_2+\theta_1) + \sin(\theta_2-\theta_1)}{\sin(\theta_2+\theta_1)}E_0$$

$$\Rightarrow \quad \underline{\underline{E_0'' = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_2+\theta_1)}E_0}}\,,$$ amplitude of the transmitted (refracted) wave.



Let the wave be now polarized in the direction parallel to the plane of incidence. Choosing the amplitudes of the magnetic field, H_0, H_0' and H_0'' towards us (perpendicular to the page plane), we have the electric field amplitudes as shown on the diagram.

The boundary condition $H_{1t} = H_{2t}$ now gives

$$H_0 + H_0' = H_0'' , \qquad (3)$$

or $$\sqrt{\varepsilon_1}\,(E_0 + E_0') = \sqrt{\varepsilon_2}\, E_0'' ,$$

or $$E_0 + E_0' = \frac{\sin\theta_1}{\sin\theta_2} E_0'' \qquad (3') .$$

The second boundary condition, $E_{1t} = E_{2t}$, yields

$$E_0 \cos\theta_1 - E_0' \cos\theta_1 = E_0'' \cos\theta_2 . \qquad (4)$$

Substituting E_0'' from (3') into (4), we have:

$$E_0 \cos\theta_1 - E_0' \cos\theta_1 = \cos\theta_2 \, \frac{\sin\theta_2}{\sin\theta_1} (E_0 + E_0')$$

$$E_0 \sin\theta_1 \cos\theta_1 - E_0' \sin\theta_1 \cos\theta_1 = \sin\theta_2 \cos\theta_2 (E_0 + E_0')$$

$$E_0 (\sin\theta_1 \cos\theta_1 - \sin\theta_2 \cos\theta_2) = (\sin\theta_1 \cos\theta_1 + \sin\theta_2 \cos\theta_2) E_0'$$

$$\Rightarrow \quad E_0' = \frac{\sin\theta_1 \cos\theta_1 + \sin\theta_2 \cos\theta_2}{\sin\theta_1 \cos\theta_1 - \sin\theta_2 \cos\theta_2} E_0' ,$$

or, using the identity provided,

$$\underline{\underline{E_0' = \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)} E_0}} , \quad \text{for the reflected wave amplitude.}$$

Substituting this into (3'), we find

$$E_0'' = \frac{\sin\theta_2}{\sin\theta_1} \left[1 + \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)} \right] E_0$$

$$= \frac{\sin\theta_2}{\sin\theta_1} \left[1 + \frac{\sin(\theta_1 - \theta_2)\cos(\theta_1 + \theta_2)}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)} \right] E_0$$

$$= \frac{\sin\theta_2}{\sin\theta_1} \, \frac{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2)\cos(\theta_1 + \theta_2)}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)} E_0$$

$$= \frac{\sin\theta_2}{\sin\theta_1} \quad \frac{\sin(\theta_1 + \cancel{\theta_2} + \theta_1 - \cancel{\theta_2})}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)} E_0$$

$$= \frac{\sin\theta_2}{\sin\theta_1} \quad \frac{2\sin\theta_1\cos\theta_1}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)} E_0$$

$$\Rightarrow \quad E_0'' = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)} E_0 \,,$$

is the amplitude of the transmitted wave

(4) (a) Electric field $\vec{E}(x,y)\, e^{i(k_g z - \omega t)}$,

Magnetic field $\vec{H}(x,y)\, e^{i(k_g z-\omega t)}$.

From Maxwell's equation $\vec{\nabla}\cdot\underbrace{\vec{E}}_{=0}$, we have:

$$\frac{\partial E_x}{\partial x} e^{i(k_g z-\omega t)} + \frac{\partial E_y}{\partial y} e^{i(k_g z-\omega t)} + E_z\, i k_g\, e^{i(k_g z-\omega t)} = 0,$$

where E_x, E_y and E_z are the components of $\vec{E}(x,y)$.

Hence,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + i k_g E_z = 0. \qquad (1).$$

Similarly, from $\vec{\nabla}\cdot\vec{H} = 0$, we find

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0, \qquad (2)$$

where H_x and H_y are the x and y components of $\vec{H}(x,y)$, and we take into account $H_z = 0$.

Considering $\vec{\nabla}\times\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$, we find the left-hand side:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x e^{i(k_g z-\omega t)} & E_y e^{i(k_g z-\omega t)} & E_z e^{i(k_g z-\omega t)} \end{vmatrix} = \text{(see next page)}$$

$$= \vec{i}\left(\frac{\partial E_z}{\partial y} e^{i(k_g z-\omega t)} - i k_g E_y e^{i(k_g z-\omega t)}\right)$$

$$-\vec{j}\left(\frac{\partial E_z}{\partial x} e^{i(k_g z-\omega t)} - i k_g E_x e^{i(k_g z-\omega t)}\right)$$

$$+\vec{k}\left(\frac{\partial E_y}{\partial x} e^{i(k_g z-\omega t)} - \frac{\partial E_x}{\partial y} e^{i(k_g z-\omega t)}\right)$$

The right-hand side is:

$$-\mu_0 \vec{i} H_x (-i\omega) e^{i(k_g z-\omega t)} - \mu_0 \vec{j} H_y (-i\omega) e^{i(k_g z-\omega t)}$$

Cancelling the exponents on both sides and equating the x, y and z components, we have:

$$\frac{\partial E_z}{\partial y} - i k_g E_y = i \mu_0 \omega H_x \qquad (3)$$

$$-\frac{\partial E_z}{\partial x} + i k_g E_x = i \mu_0 \omega H_y \qquad (4)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \qquad (5)$$

Similarly, using $\vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$, we find:

$$-i k_g H_y = -i \varepsilon_0 \omega E_x \qquad (6)$$

$$i k_g H_x = -i \varepsilon_0 \omega E_y \qquad (7)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i \varepsilon_0 \omega E_z \qquad (8)$$

From (7), $H_x = -\frac{\varepsilon_0 \omega}{k_g} E_y$, and using this in (3) gives:

$$\frac{\partial E_z}{\partial y} - i k_g E_y = i \mu_0 \omega \left(-\frac{\varepsilon_0 \omega}{k_g} E_y\right)$$

$$\frac{\partial E_z}{\partial y} = \frac{i}{k_g}\left(k_g^2 - \mu_0 \varepsilon_0 \omega^2\right) E_y$$

$$\Rightarrow \qquad E_y = \frac{k_g}{i(k_g^2 - \varepsilon_0\mu_0\omega^2)} \frac{\partial E_z}{\partial y} . \qquad (9)$$

Similarly, from (6), $H_y = \frac{\varepsilon_0 \omega}{k_g} E_x$, and using this in (4), we find:

$$-\frac{\partial E_z}{\partial x} + i k_g E_x = i \mu_0 \omega \frac{\varepsilon_0 \omega}{k_g} E_x$$

$$-\frac{\partial E_z}{\partial x} = -\frac{i}{k_g}\left(k_g^2 - \varepsilon_0\mu_0\omega^2\right) E_x$$

$$\Rightarrow \qquad E_x = \frac{k_g}{i(k_g^2 - \varepsilon_0\mu_0\omega^2)} \frac{\partial E_z}{\partial x} . \qquad (10)$$

Using these, we obtain for the magnetic field components:

$$H_x = -\frac{\varepsilon_0\omega}{k_g} E_y = -\frac{\varepsilon_0\omega}{i(k_g^2 - \varepsilon_0\mu_0\omega^2)} \frac{\partial E_z}{\partial y} ,$$

and

$$H_y = \frac{\varepsilon_0\omega}{k_g} E_x = \frac{\varepsilon_0\omega}{i(k_g^2 - \varepsilon_0\mu_0\omega^2)} \frac{\partial E_z}{\partial x} .$$

Note that because of the relations between H_x and E_y, and between H_y and E_x, equations (2) and (5) are equivalent, and so are (1) and (8).

(b) To find the differential equation satisfied by E_z, we substitute (9) and (10) into (1), which gives:

$$\frac{k_g}{i(k_g^2 - \varepsilon_0\mu_0\omega^2)} \left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2}\right) + i k_g E_z = 0 ,$$

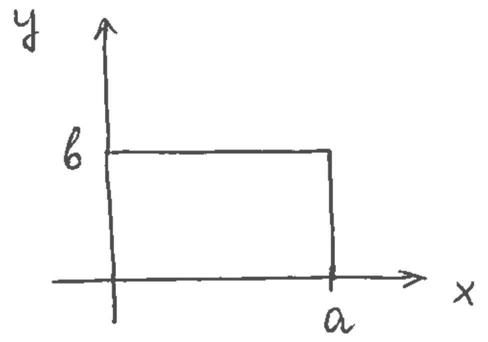
or

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (\varepsilon_0\mu_0\omega^2 - k_g^2) E_z = 0 .$$

Introducing the speed of light $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$, we can re-write this equation as

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k_g^2\right) E_z = 0. \qquad (11)$$

(c) The electric field inside the conducting walls is zero. This means that inside the waveguide, near the walls, we must have $E_t = 0$, where "t" indicates the tangential component. This means that we must have $E_z = 0$ at $x = 0$, $x = a$, $y = 0$ and $y = b$.



The solution

$$E_z(x,y) = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (12)$$

with integer m and n indeed satisfies this, since $\sin 0 = 0$ and $\sin m\pi = \sin n\pi = 0$.

For $x = 0$ and $x = a$ we must also have $E_y = 0$.

From (9), $E_y \propto \frac{\partial E_z}{\partial y}$ (is proportional to).

$$\frac{\partial E_z}{\partial y} = A \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$ indeed vanishes

for $x = 0$ and $x = a$.

Similarly, we must have $E_x = 0$ for $y = 0$ and $y = b$. From equation (10),

$$E_x \propto \frac{\partial E_z}{\partial x} = A \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

which vanishes at $y = 0$ and at $y = b$, as required.

Finally, substituting (12) into (11), we obtain:

$$-A\left(\frac{m\pi}{a}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} - A\left(\frac{n\pi}{b}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$+\left(\frac{\omega^2}{c^2}-k_g^2\right) A \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0,$$

which must be valid for all x and y, so that

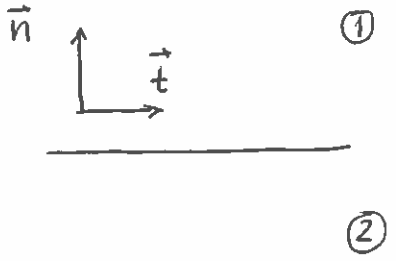
$$\left[-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2+\frac{\omega^2}{c^2}-k_g^2\right] A \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0$$

gives $[\dots]=0$, or equivalently,

$$\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 = \frac{\omega^2}{c^2}-k_g^2 .$$

This equations determines the allowed values of the wavenumber k_g, for a given frequency ω, in terms of the integers m and n and the dimensions of the waveguide a and b.

5 (a) Let unit vector $\vec{n}$ be normal to the interface between the two media, and unit vector $\vec{t}$ tangential to it. The boundary conditions then are:



$$E_{1t} = E_{2t},$$

$$D_{1n} - D_{2n} = \sigma,$$

$$B_{1n} = B_{2n},$$

$$(\vec{H}_1-\vec{H}_2)\cdot\vec{t} = \vec{J}\cdot(\vec{n}\times\vec{t}),$$

where σ is the surface density of free charges, $\vec{J}$ is the surface current density, and the indices 1 and 2 refer to the fields in media 1 and 2, respectively.

Note that the last of the boundary conditions can be written as $(\vec{H}_1 - \vec{H}_2)\cdot\vec{t} = (\vec{J}\times\vec{n})\cdot\vec{t}$.

(b) i. The electric field: $\vec{E}(\rho,\varphi)\, e^{i(kz-\omega t)}$,

the magnetic field: $\vec{H}(\rho,\varphi)\, e^{i(kz-\omega t)}$.

Using $\vec{\nabla}\times\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$, we find (using cylindrical coordinates and components):

$$\begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\vec{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ E_\rho e^{i(kz-\omega t)} & \rho E_\varphi e^{i(kz-\omega t)} & E_z e^{i(kz-\omega t)} \end{vmatrix} = \frac{1}{\rho}\hat{\rho}\left(\frac{\partial E_z}{\partial\varphi} e^{i(kz-\omega t)} - \rho E_\varphi ik\, e^{i(kz-\omega t)}\right)$$

$$-\hat{\varphi}\left(\frac{\partial E_z}{\partial\rho} e^{i(kz-\omega t)} - E_\rho ik\, e^{i(kz-\omega t)}\right) + \frac{1}{\rho}\vec{k}\left(\frac{\partial(\rho E_\varphi)}{\partial\rho} - \frac{\partial E_\rho}{\partial\varphi}\right)e^{i(kz-\omega t)}$$

The right-hand side is

$$-\mu_0(\hat{\rho} H_\rho + \hat{\varphi} H_\varphi)(-i\omega)\, e^{i(kz-\omega t)}.$$

$H_z = 0$, so no $\vec{k}$ term.

Comparing the $\hat{\rho}$ and $\hat{\varphi}$ components, we have:

$$\frac{1}{\rho}\frac{\partial E_z}{\partial\varphi} - ikE_\varphi = i\mu_0\omega H_\rho \qquad (1)$$

$$-\frac{\partial E_z}{\partial\rho} + ikE_\rho = i\mu_0\omega H_\varphi \qquad (2)$$

Similarly, from $\vec{\nabla}\times\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$, we obtain:

$$-ikH_\varphi = -i\varepsilon_0\omega E_\rho \qquad (3)$$

$$ikH_\rho = -i\varepsilon_0\omega E_\varphi, \qquad (4)$$

with no H_z terms on the left-hand side, since $H_z = 0$.

From (3), $H_\varphi = \dfrac{\varepsilon_0 \omega}{k} E_\rho$, and substituting this into (2), we find:

$$-\frac{\partial E_z}{\partial \rho} + ikE_\rho = i\,\frac{\mu_0 \varepsilon_0 \omega^2}{k} E_\rho$$

$$-\frac{\partial E_z}{\partial \rho} = -\frac{i}{k}\left(k^2 - \varepsilon_0 \mu_0 \omega^2\right) E_\rho$$

$$\Rightarrow \quad \underline{\underline{E_\rho = \frac{k}{i(k^2 - \varepsilon_0 \mu_0 \omega^2)} \frac{\partial E_z}{\partial \rho}}} . \qquad (5)$$

From (4), $H_\rho = -\dfrac{\varepsilon_0 \omega}{k} E_\varphi$, and substituting into (1), we have:

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} - ikE_\varphi = -i\,\frac{\varepsilon_0 \mu_0 \omega^2}{k} E_\varphi$$

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} = \frac{i}{k}\left(k^2 - \varepsilon_0 \mu_0 \omega^2\right) E_\varphi$$

$$\Rightarrow \quad \underline{\underline{E_\varphi = \frac{k}{i(k^2 - \varepsilon_0 \mu_0 \omega^2)} \frac{1}{\rho}\frac{\partial E_z}{\partial \varphi}}} . \qquad (6)$$

Consequently,

$$\underline{\underline{H_\rho = -\frac{\varepsilon_0 \omega}{i(k^2 - \varepsilon_0 \mu_0 \omega^2)} \frac{1}{\rho}\frac{\partial E_z}{\partial \varphi}}} ,$$

and

$$\underline{\underline{H_\varphi = \frac{\varepsilon_0 \omega}{i(k^2 - \varepsilon_0 \mu_0 \omega^2)} \frac{\partial E_z}{\partial \rho}}} .$$

ii. To find the equation satisfied by E_z, we can use Maxwell's equation $\vec{\nabla} \cdot \vec{E} = 0$, which gives:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho E_\rho e^{i(kz-\omega t)}\right) + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\left(E_\varphi e^{i(kz-\omega t)}\right) + \frac{\partial}{\partial z}\left(E_z e^{i(kz-\omega t)}\right) = 0 .$$

Since E_z does not depend on z, this gives:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\psi}{\partial\psi} + ikE_z = 0.$$

Substituting E_ρ and E_ψ from (5) and (6), we obtain:

$$\frac{k}{i(k^2-\varepsilon_0\mu_0\omega^2)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\psi^2}\right] + ikE_z = 0$$

$$\underline{\underline{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\psi^2} + \left(\frac{\omega^2}{c^2}-k^2\right)E_z = 0,}}$$

where we have introduced the speed of light, $\varepsilon_0\mu_0 = \frac{1}{c^2}$.

Note that the first two terms on the left-hand side are the ρ and ψ parts of ∇^2 in cylindrical coordinates. The above equation could in fact be obtained from the wave equation satisfied by the electric field:

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0.$$

Applying this to $E_z(\rho,\psi)\,e^{i(kz-\omega t)}$ and using

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial z^2},$$

we obtain:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right)e^{i(kz-\omega t)} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\psi^2}e^{i(kz-\omega t)} - k^2E_z e^{i(kz-\omega t)}$$

$$- \frac{1}{c^2}(-\omega^2)E_z e^{i(kz-\omega t)} = 0,$$

which, upon cancelling the exponents, gives equation (7).

(6) Electric and magnetic fields

$$\vec{\mathcal{E}}(\vec{r},t) = \vec{E}\, e^{i(kz-\omega t)}$$

and $$\vec{\mathcal{H}}(\vec{r},t) = \vec{H}\, e^{i(kz-\omega t)},$$

in which $\vec{E}$ and $\vec{H}$ do not depend on z (the coordinate along the waveguide) and $H_z = 0$, are the same as in questions 4 and 5, so we could use some of the results obtained there to solve this problem. It is instructive, however, to proceed in a slightly different, more general, way

(a) Let us write all vectors involved in terms of their components along the z axis, and the transversal part (which we donote using subscript t), which lies in the plane perpendicular to z.

Hence:
$$\left.\begin{aligned} \vec{E} &= \vec{E}_t + \vec{e}_z E_z \\ \vec{\nabla} &= \vec{\nabla}_t + \vec{e}_z \frac{\partial}{\partial z} \end{aligned}\right\} \text{ Here } \vec{e}_z \text{ is the unit vector along } z$$
$$\vec{H} = \vec{H}_t \quad (\text{since } H_z = 0).$$

i. Maxwell's equation
$$\vec{\nabla}\times\vec{\mathcal{H}} = \varepsilon\,\frac{\partial\vec{\mathcal{E}}}{\partial t} \quad \Bigg\} \text{ Here } \vec{\mathcal{H}} \text{ and } \vec{\mathcal{E}} \text{ are "complete" fields}$$

then gives:
$$\left(\vec{\nabla}_t + \vec{e}_z\frac{\partial}{\partial z}\right)\times\vec{H}_t e^{i(kz-\omega t)} = \varepsilon\left(\vec{E}_t + \vec{e}_z E_z\right)(-i\omega)\, e^{i(kz-\omega t)}$$

$$\vec{\nabla}_t\times\vec{H}_t e^{i(kz-\omega t)} + \vec{e}_z\times\vec{H}_t\,(ik)\, e^{i(kz-\omega t)}$$
$$= -i\omega\varepsilon\left(\vec{E}_t + \vec{e}_z E_z\right) e^{i(kz-\omega t)}.$$

Cancelling the exponents, we have:

$$\underbrace{\vec{\nabla}_t \times \vec{H}_t}_{} + ik\, \vec{e}_z \times \vec{H}_t = -i\varepsilon\omega\, \vec{E}_t - i\omega\varepsilon\, \vec{e}_z E_z .$$

This vector is in the direction perpendicular to the transversal plane, i.e., along $\vec{e}_z$.

Multiplying the above equation in a cross product with $\vec{e}_z$, we make this term, as well as the last one on the right-hand side, vanish ($\vec{e}_z \times \vec{e}_z = 0$).

Hence,

$$ik\, \vec{e}_z \times (\vec{e}_z \times \vec{H}_t) = -i\varepsilon\omega\, \vec{e}_z \times \vec{E}_t$$

$$ik \left[\vec{e}_z \underbrace{(\vec{e}_z \cdot \vec{H}_t)}_{=0} - \vec{H}_t \underbrace{(\vec{e}_z \cdot \vec{e}_z)}_{=1} \right] = -i\varepsilon\omega\, \vec{e}_z \times \vec{E}_t .$$

$$\Rightarrow \quad -ik\, \vec{H}_t = -i\varepsilon\omega\, \vec{e}_z \times \vec{E}_t$$

$$\Rightarrow \quad \vec{H}_t = \frac{\varepsilon\omega}{k}\, \vec{e}_z \times \vec{E}_t . \qquad (1)$$

ii. Maxwell's 3rd equation, $\vec{\nabla} \times \vec{\mathcal{E}} = -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t}$

gives:

$$\left(\vec{\nabla}_t + \vec{e}_z \frac{\partial}{\partial z}\right) \times (\vec{E}_t + \vec{e}_z E_z)\, e^{i(kz-\omega t)} = -\mu \frac{\partial}{\partial t} \vec{H}_t e^{i(kx-\omega t)}$$

$$\left(\vec{\nabla}_t \times \vec{E}_t + \vec{e}_z \times \vec{E}_t (ik) - \vec{e}_z \times \vec{\nabla}_t E_z \right) e^{i(kz-\omega t)}$$
$$= +i\mu\omega\, \vec{H}_t\, e^{i(kx-\omega t)}$$

Cancelling the exponents, we have:

$$\underbrace{\vec{\nabla}_t \times \vec{E}_t}_{\text{This vector is along } \vec{e}_z} + ik\, \vec{e}_z \times \vec{E}_t - \vec{e}_z \times \vec{\nabla}_t E_z = i\mu\omega\, \vec{H}_t$$

Multiplying this equation in a cross product with $\vec{e}_z$ makes the first term on the left-hand side vanish, and we have:

$$ik\,\vec{e}_z \times (\vec{e}_z \times \vec{E}_t) - \vec{e}_z \times (\vec{e}_z \times \vec{\nabla}_t E_z) = i\mu\omega\,\vec{e}_z \times \vec{H}_t$$

$$ik\left[\vec{e}_z \underbrace{(\vec{e}_z \cdot \vec{E}_t)}_{=0} - \vec{E}_t \underbrace{(\vec{e}_z \cdot \vec{e}_z)}_{=1}\right] - \left[\vec{e}_z \underbrace{(\vec{e}_z \cdot \vec{\nabla}_t E_z)}_{=0} - \vec{\nabla}_t E_z \underbrace{(\vec{e}_z \cdot \vec{e}_z)}_{=1}\right]$$

since $\vec{\nabla}_t E_t$ is a vector in the plane $\perp$ to $\vec{e}_z$

$$= i\mu\omega\,\underbrace{\vec{e}_z \times (\vec{e}_z \times \vec{E}_t)\,\frac{\varepsilon\omega}{k}}$$

Here we have used the result from part i.

$$\Rightarrow \quad -ik\vec{E}_t + \vec{\nabla}_t E_z = -i\frac{\varepsilon\mu\omega^2}{k}\vec{E}_t$$

$$\vec{e}_z \underbrace{(\vec{e}_z \cdot \vec{E}_t)}_{=0} - \vec{E}_t \underbrace{(\vec{e}_z \cdot \vec{e}_z)}_{=1}$$

$$\vec{\nabla}_t E_z = -\frac{i}{k}\left(\varepsilon\mu\omega^2 - k^2\right)\vec{E}_t ,$$

or

$$\underline{\underline{-\frac{iK^2}{k}\vec{E}_t = \vec{\nabla}_t E_z}} , \qquad (2)$$

where $K^2 = \varepsilon\mu\omega^2 - k^2$.

This equation (2), together with (1), shows that the transversal parts of the electric and magnetic fields ($\vec{E}_t$ and $\vec{H}_t$) are functions of E_z.

iii. From Maxwell's equation $\vec{\nabla} \cdot \vec{E} = 0$,

we have: $\left(\vec{\nabla}_t + \vec{e}_z \frac{\partial}{\partial z}\right) \cdot \left[(\vec{E}_t + \vec{e}_z E_z)\, e^{i(kz-\omega t)}\right] = 0.$

$$\vec{\nabla}_t \cdot \vec{E}_t\, \cancel{e^{i(kz-\omega t)}} + E_z\, ik\, \cancel{e^{i(kz-\omega t)}} = 0 .$$

Substituting $\vec{E}_t$ from (2), we have:

$$\vec{\nabla}_t \cdot \left(\frac{ik}{K^2}\vec{\nabla}_t E_z\right) + ikE_z = 0 .$$

$$\vec{\nabla}_t \cdot \vec{\nabla}_t E_z + K^2 E_z = 0 ,$$

or $$(\vec{\nabla}_t^2 + K^2) E_z = 0 ,$$

which is the equation satisfied by E_z.

(b) The solution for a cylindrical waveguide $(0 \le \rho \le a)$ is

$$E_z(\rho, \varphi) = \alpha J_n(K\rho) \cos(n\varphi) , \quad n = 0, 1, \dots$$

On the conducting wall of the waveguide we must have $E_z = 0$ (vanishing of the tangential component). Hence we require

$$Ka = z_{n,m} ,$$

where $z_{n,m}$ is the mth root of $J_n(z)$ $(m = 1, 2, \dots)$

Using the expression for K from part (a), we have:

$$\sqrt{\varepsilon\mu\omega^2 - k^2}\, a = z_{n,m} .$$

Squaring this equation gives:

$$(\varepsilon\mu\omega^2 - k^2) a^2 = z_{n,m}^2$$

$$\varepsilon\mu\omega^2 - k^2 = \frac{z_{n,m}^2}{a^2}$$

$$k^2 = \varepsilon\mu \left(\omega^2 - \frac{z_{n,m}^2}{\varepsilon\mu a^2} \right) .$$

Denoting $\omega_{nm} = \frac{z_{n,m}}{\sqrt{\varepsilon\mu}\, a}$, we have:

$$k^2 = \varepsilon\mu (\omega^2 - \omega_{nm}^2) , \text{ as required.}$$

Since $k^2 \ge 0$, waves with frequencies ω smaller than ω_{nm} cannot propagate in the waveguide. Hence,

$$\omega_{nm} = \frac{z_{n,m}}{\sqrt{\varepsilon\mu}\, a}$$ is called the "cut-off frequency".

Each pair of integers n and m corresponds to a particular shape of the solution $E_z(\rho, \varphi)$.

Such shapes (and the corresponding indices n and m) are known as modes.