AMA3006 Problem sheet 1

Examples

(1) $$\frac{d^2y}{dx^2} + \lambda y = 0 , \qquad y'(0) = y(l) = 0 .$$

This is a 2nd-order linear differential equation with constant coefficients. To solve it, seek solution in the form

$$y = e^{sx}$$

$$y' = s e^{sx} , \quad y'' = s^2 e^{sx}$$

Substitute into the equation:

$$s^2 e^{sx} + \lambda e^{sx} = 0 \qquad \Big\} \text{ multiply by } e^{-sx}$$

$$s^2 + \lambda = 0 \quad : \quad \underline{\text{auxiliary equation}}.$$

$$s^2 = -\lambda$$

$$s = \pm\sqrt{-\lambda} .$$

If $\lambda \neq 0$, the auxiliary equation has two distinct roots, and the general solution of the DE is:

$$y = C_1 e^{\sqrt{-\lambda}\, x} + C_2 e^{-\sqrt{-\lambda}\, x} ,$$

where C_1 and C_2 are arbitrary constants.

To implement the boundary conditions,

$$y' = \sqrt{-\lambda}\, C_1 e^{\sqrt{-\lambda}\, x} - C_2 \sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\, x}$$

$y'(0) = 0$ gives: $\sqrt{-\lambda}\,(C_1 - C_2) = 0$

$$\Rightarrow \quad C_2 = C_1 \equiv C . \qquad (\lambda \neq 0).$$

So $y = C\left(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}\right)$

Using the 2nd boundary condition $y(\ell)=0$:

$$C\left(e^{\sqrt{-\lambda}\ell} + e^{-\sqrt{-\lambda}\ell}\right) = 0$$

For $\lambda < 0$ the sum of two real exponents cannot be zero $\Rightarrow$ $C = 0$ and the equation has only a trivial solution $y = 0$.

Consider $\lambda > 0$. Then $\sqrt{-\lambda} = i\sqrt{\lambda}$

and

$$y = C\left(e^{i\sqrt{\lambda}x} + e^{-i\sqrt{\lambda}x}\right)$$

$$y = \underbrace{2C}_{\text{rename } C}\cos(\sqrt{\lambda}x)$$

$$y = C\cos(\sqrt{\lambda}x)$$

Recall:

$e^{i\theta} = \cos\theta + i\sin\theta$

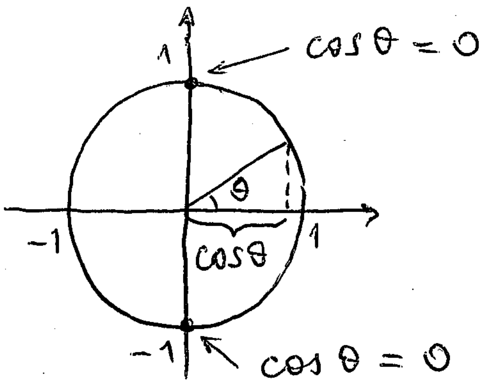
$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Now, the boundary condition requires:

$$C\cos(\sqrt{\lambda}\ell) = 0 \qquad (*)$$



$\cos\theta = 0$ for

$$\theta = \frac{\pi}{2} + \pi n, \quad n \in \mathbb{Z}.$$

Hence, $(*)$ gives:

$$\sqrt{\lambda}\ell = \frac{\pi}{2} + \pi n, \quad n = 0, 1, \ldots$$

$$\sqrt{\lambda} = \frac{\pi(n+\frac{1}{2})}{\ell} \qquad (\sqrt{\lambda}\ell > 0).$$

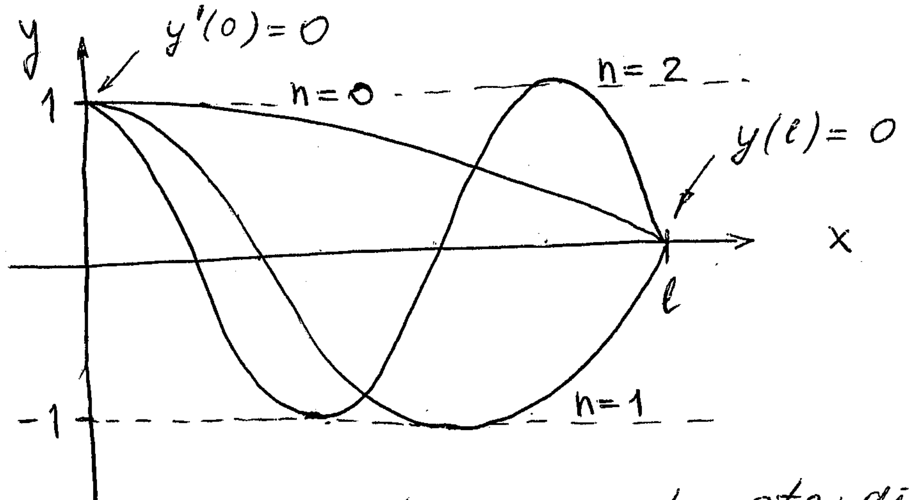
$$\boxed{\lambda = \frac{\pi^2(n+\frac{1}{2})^2}{\ell^2}}$$

Values of λ for which the equation has nontrivial solutions.

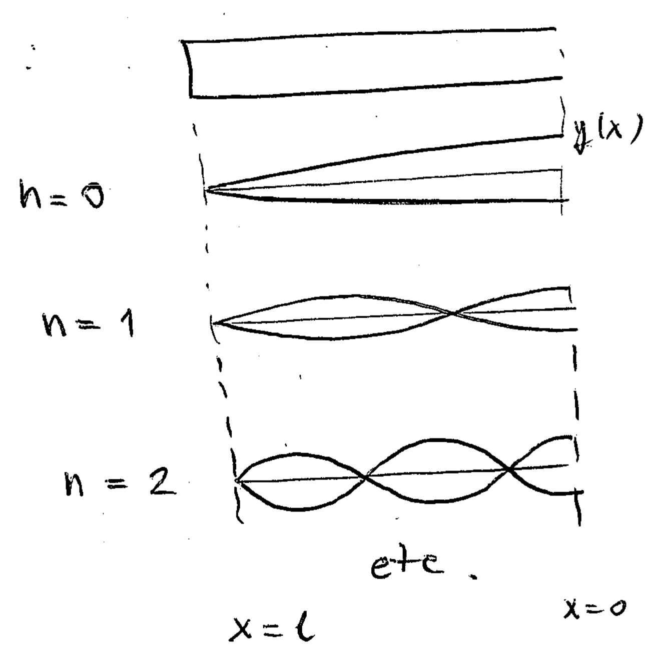
The corresponding solutions are:

$$y = C \cos\left[\pi\left(n+\tfrac{1}{2}\right)\frac{x}{\ell}\right]$$

We can sketch the first few (for $C=1$)



These solutions in fact represent standing waves, e.g., those that occur when air vibrates in a semiclosed (organ) pipe:



$y(x)$ then represents the displacement of air along the pine.

Naturally $y(\ell) = 0$ (at the closed end)

and y is a maximum ($y'(0) = 0$) near the open end, ② $u = f(x^2 - y^2)$ (*)

By taking derivatives of u, with respect to x and y, we should be able to eliminate f and obtain a partial DE, for which (*) is the solution.

(1) $u_x \equiv \frac{\partial u}{\partial x} = f'(x^2-y^2) \cdot 2x$ } using chain rule

[f' is the derivative of f with respect to its argument]

(2) $u_y \equiv \frac{\partial u}{\partial y} = f'(x^2-y^2)(-2y)$

Dividing (1) by (2), we have:

$$\frac{u_x}{u_y} = -\frac{2x}{2y}$$

$$y u_x = -x u_y$$

$$\underline{y u_x + x u_y = 0}.$$

③ $u_{xxx} + u_x = 0$ for $u(x, y, z)$

Integrating this equation with respect to x, we have:

$$u_{xx} + u = f(y, z) \qquad (*)$$

[The right-hand side is an arbitrary "constant" (with respect to x), i.e. an arbitrary function of y and z.]

(*) is a linear inhomogeneous equation.

Solving the homogeneous version first,

$$u_{xx} + u = 0,$$

we have:

$$u = g(y,z)\cos x + h(y,z)\sin x$$

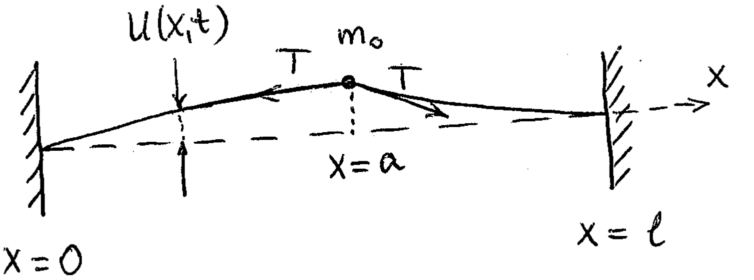
arbitrary functions of y and z (i.e. "constants" with respect to x).

To solve the inhomogeneous equation $(*)$, it sufficient to add $f(y,z)$ to the above solution:

$$u(x,y,z) = f(y,z) + g(y,z)\cos x + h(y,z)\sin x.$$

(one can check that this is correct by back-substitution into $(*)$.)

(4)



Boundary conditions:

$$u(0,t) = 0$$
$$u(\ell,t) = 0$$

$u(x,t)$ - displacement of the string.

For $0 \le x < a$ and $a < x \le \ell$ the wave equation is exactly the same as that for a uniform string,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (*)$$

$$c = \sqrt{\frac{T}{\rho}}$$

T - tension

ρ - linear mass density

Force acting on m_0: $T\sin\theta\big|_{x=a+0} - T\sin\theta\big|_{x=a-0}$

Here $x = a+0$ and $x = a-0$ mean values of x taken "just to the right" and "just to the left" of a. The angle θ is that the string makes with the x axis. For small θ, $\theta \ll 1$,

$$\sin\theta \simeq \tan\theta = \frac{\partial u}{\partial x}.$$

Hence, the force acting on m_0 is $T\left(\frac{\partial u}{\partial x}\Big|_{x=a+0} - \frac{\partial u}{\partial x}\Big|_{x=a-0}\right)$

Newton's 2nd law for m_0:

$$m_0 \frac{\partial^2 u}{\partial t^2}\Big|_{x=a} = T\left(\frac{\partial u}{\partial x}\Big|_{x=a+0} - \frac{\partial u}{\partial x}\Big|_{x=a-0}\right) \quad (**).$$

Therefore, in order to solve the problem, one needs to find a solution of (*) for $0 \le x \le a$ satisfying $u(0,t) = 0$; let's call it $u_1(x,t)$. Then do the same for $a \le x \le l$ with $u(l,t) = 0$; call this solution $u_2(x,t)$. The two solutions must match at $x = a$:

$$u_1(a,t) = u_2(a,t),$$

and (**) must be satisfied:

$$\frac{m_0}{T} \frac{\partial^2 u}{\partial t^2}\Big|_{x=a} = \frac{\partial u_2}{\partial x}\Big|_{x=a} - \frac{\partial u_1}{\partial x}\Big|_{x=a}.$$