AMA 3006 Problem sheet 1

Homework problems

SOLUTIONS

(1) (a) $$\frac{dx}{dt} = -\lambda x, \quad \lambda = \text{const}, \quad x(0) = x_0.$$

$$\frac{dx}{x} = -\lambda \, dt$$

$$\int \frac{dx}{x} = -\lambda \int dt$$

$$\ln|x| = -\lambda t + C$$

$$|x| = e^C e^{-\lambda t}$$

$$x = \underbrace{\pm e^C}_{\text{rename } C} e^{-\lambda t}$$

$$x = C e^{-\lambda t}$$

Using $x(0) = x_0$:

$$x_0 = C e^0 \Rightarrow C = x_0$$

$$\Rightarrow \quad \underline{x(t) = x_0 e^{-\lambda t}}.$$

(b) $$y' + y \tan x = \cos x.$$

This is a 1st-order linear differential equation. It can be solved by either of the two methods: method of a varying constant, or the integrating factor method. Let's use the former.

<u>Step 1</u>: solve the homogeneous equation

$$\frac{dy}{dx} + y \tan x = 0$$

$$\frac{dy}{dx} = -y \tan x$$

$$\frac{dy}{y} = -\tan x \, dx$$

$$\int \frac{dy}{y} = -\int \frac{\sin x \, dx}{\cos x} \qquad -\sin x \, dx = d(\cos x)$$

$$\ln y = \int \frac{d(\cos x)}{\cos x}$$

$$\ln y = \ln(\cos x) + C$$

$$y = \underbrace{e^{c}}_{\text{rename } C} e^{\ln(\cos x)}$$

$$y = C \cos x$$

Step 2 : replace constant C with function $C(x)$

$$y = C(x)\cos x$$

Substitute into the original equation:

$$c'(x)\cos x - \cancel{C(x)\sin x} + \cancel{C(x)\cos x \frac{\sin x}{\cos x}} = \cos x$$

$$\frac{dC}{dx}\cos x = \cos x$$

$$\frac{dC}{dx} = 1$$

$$\int dC = \int dx$$

$$C(x) = x + C_1$$

Therefore $$y = (x + C_1)\cos x$$

$$y = \underbrace{x\cos x}_{\text{particular integral}} + \underbrace{C_1 \cos x}_{\text{complementary function}}$$

(c) $y'' + 4y' + 5y = \sin 2x$

This is a 2nd-order linear DE. Solving the homogeneous equation first:

$$y'' + 4y' + 5y = 0 . \qquad (*)$$

Seek solution in the form

$$y = e^{sx}$$

$$y' = se^{sx}, \quad y'' = s^2 e^{sx}$$

Substituting back into $(*)$:

$$s^2 e^{sx} + 4se^{sx} + 5e^{sx} = 0 \quad \Big\} \times e^{-sx}$$

$$s^2 + 4s + 5 = 0 \quad \text{- auxiliary equation}$$

Roots: $$s = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2}$$

$$\Rightarrow \qquad s = -2 \pm i$$

The general solution of the homogeneous equation is

$$y = C_1 e^{(-2+i)x} + C_2 e^{(-2-i)x},$$

which can also be written in the real form

$$y = e^{-2x}(C_1 \cos x + C_2 \sin x)$$

(with a different choice of C_1, C_2).

To solve the inhomogeneous equation, note that the function on the R.H.S. has the form

$$e^{ax}[P_n(x)\cos bx + Q_m(x)\sin bx]$$

with $a = 0$, $P_n(x) = 0$, $Q_m(x) = 1$, $b = 2$.

Hence, we can seek particular integral in the form:

$$y_p = A\cos 2x + B\sin 2x$$

[$a \pm ib = \pm 2i$ is not among the roots of the auxiliary equation; A and B are polynomials of degree zero (as $Q_m(x)$), with undetermined coefficients, i.e. A and B.]

$$y_p' = -2A\sin 2x + 2B\cos 2x$$

$$y_p'' = -4A\cos 2x - 4B\sin 2x$$

Substituting back into the differential equation:

$$-4A\cos 2x - 4B\sin 2x + 4(-2A\sin 2x + 2B\cos 2x)$$

$$+5(A\cos 2x + B\sin 2x) = \sin 2x$$

$$(-4A + 8B + 5A)\cos 2x + (-4B - 8A + 5B)\sin 2x = \sin 2x$$

$$(A + 8B)\cos 2x + (B - 8A)\sin 2x = \sin 2x$$

$$\begin{cases} A + 8B = 0 \\ B - 8A = 1 \end{cases} \Rightarrow A = -8B$$

substitute into 2nd eq.

$$B + 64B = 1 \iff 65B = 1$$

$$\Rightarrow B = \frac{1}{65}, \quad A = -\frac{8}{65}$$

$$\Rightarrow y_p = -\frac{8}{65}\cos 2x + \frac{1}{65}\sin 2x$$

So, the general solution of the inhomogeneous equation is:

$$y = \underbrace{C_1 e^{-2x}\cos x + C_2 e^{-2x}\sin x}_{\text{complementary function}} \underbrace{- \frac{8}{65}\cos 2x + \frac{1}{65}\sin 2x}_{\text{particular integral}}$$

② $\frac{d^2y}{dx^2} + \lambda y = 0$, $y(0) = y(1) = 0$.

This is a linear 2nd-order differential equation with constant coefficients. We seek its solution in the form:

$$y = e^{sx}$$

$y' = se^{sx}$, $y'' = s^2 e^{sx}$ – substituting:

$$s^2 e^{sx} + \lambda e^{sx} = 0 \quad \Big\} \times e^{-sx}$$

$$s^2 + \lambda = 0$$

$$s^2 = -\lambda$$

$$s = \pm\sqrt{-\lambda}$$

General solution:

$$y = C_1 e^{\sqrt{-\lambda}\,x} + C_2 e^{-\sqrt{-\lambda}\,x} \quad \Big\} \text{ we assume } \lambda \neq 0$$

Implementing the boundary condition:

$y(0) = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_2 = -C_1 \equiv -C$

$$y = C\left(e^{\sqrt{-\lambda}\,x} - e^{-\sqrt{-\lambda}\,x}\right)$$

2nd boundary condition $y(1) = 0$

$$C\left(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}\right) = 0 \qquad (*)$$

For $\lambda < 0$ the exponents are real, and $e^{\sqrt{-\lambda}} > 1$, while $e^{-\sqrt{-\lambda}} < 0$. So $(*)$ can only be satisfied for $C = 0$ and we only have a trivial solution $y = 0$.

For $\lambda > 0$ $\qquad \sqrt{-\lambda} = i\sqrt{\lambda}$,

so $\qquad y = C\left(e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}\right),$

which can be written as

$$y = C\sin(\sqrt{\lambda}x)$$

(with a different C).

Recall:

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

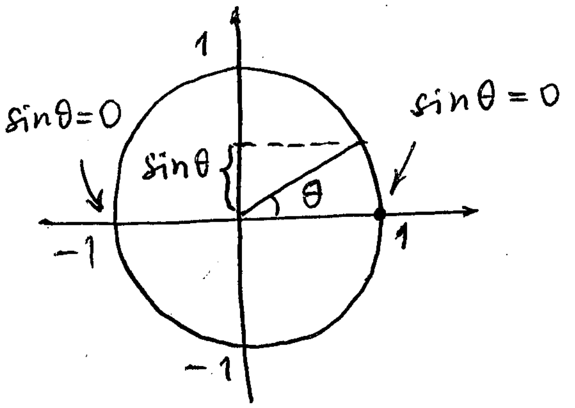
Now, $y(1) = 0$ gives

$$C\sin(\sqrt{\lambda}) = 0$$

So: $\qquad \sqrt{\lambda} = \pi n$

$(n = 1, 2, \dots)$

$$\boxed{\lambda = \pi^2 n^2}$$



$\sin\theta = 0$ for

$\theta = \pi n,\ n \in \mathbb{Z}$

For these values of λ the differential equation

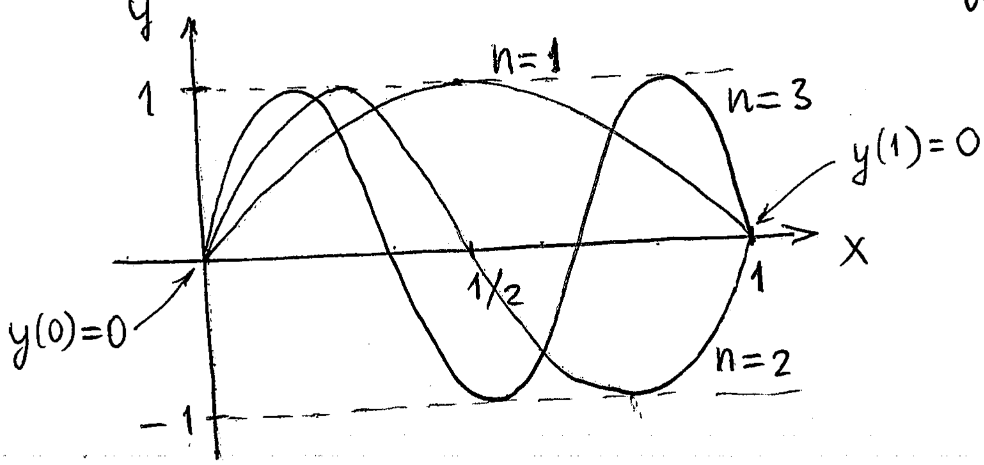
$y'' + \lambda y = 0$ with boundary conditions $y(0) = y(1) = 0$

has nontrivial solutions:

$$y = C\sin(\pi n x)$$

Sketching the first few (for $0 \le x \le 1$)

We use $C = 1$.



(3) (a) $u_x u_y - 3u = x$ 1st order, nonlinear

(b) $y\, u_{xxy} - e^x u_x + 3 = 0$ 3rd order, linear inhomogeneous

(c) $\rho u_{tt} + \mu u_{xxxx} = 0$ 4th order, linear homogeneous

(d) $u u_{xy} - u_x u_y = 0$ 2nd order, nonlinear

(e) $u_{xx} + u_{yy} + \log u = \log x$ 2nd order, nonlinear (because of $\log u$ term)

(4) $u = f(x^2+y^2)$

$$u_x = f'(x^2+y^2)\,\frac{\partial(x^2+y^2)}{\partial x}$$ } using chain rule

(1) $$u_x = f'(x^2+y^2)\, 2x$$

$$u_y = f'(x^2+y^2)\,\frac{\partial(x^2+y^2)}{\partial y}$$

(2) $$u_y = f'(x^2+y^2)\, 2y$$

Dividing (1) by (2) to eliminate f', we obtain

$$\frac{u_x}{u_y} = \frac{2x}{2y}$$

$$y u_x = x u_y$$

$$\underline{y u_x - x u_y = 0}$$, as required.

(5) (a) $u = f(xy)$

(1) $$u_x = f'(xy)\,\frac{\partial}{\partial x}(xy) = f'(xy)\, y$$

(2) $$u_y = f'(xy)\,\frac{\partial}{\partial y}(xy) = f'(xy)\, x$$

Dividing (1) by (2) to eliminate $f'(xy)$:

$$\frac{u_x}{u_y} = \frac{y}{x} \iff \underline{x u_x - y u_y = 0}.$$

(b) $u = f(x^2+y^2) + g(x)$

$$u_x = f'(x^2+y^2)\,\frac{\partial(x^2+y^2)}{\partial x} + g'(x)$$

$$u_x = f'(x^2+y^2)\,2x + g'(x) \qquad (1)$$

$$u_y = f'(x^2+y^2)\,2y \quad \Rightarrow \quad f'(x^2+y^2) = \frac{u_y}{y}$$

Substituting this into (1) to eliminate $f'(x^2+y^2)$:

$$u_x = \frac{x u_y}{y} + g'(x) \qquad (2)$$

To get rid of $g'(x)$, differentiate (2) with respect to y:

$$u_{xy} = x\,\frac{u_{yy}}{y} - x\,\frac{u_y}{y^2} \quad \Big\} \times y^2$$

$$y^2 u_{xy} = xy\,u_{yy} - x u_y,$$

or $\underline{y^2 u_{xy} - xy\,u_{yy} + x u_y = 0\,.}$

6 (a) $u_{xxy} = 1$

Integrate with respect to y:

$$\int u_{xxy}\,dy = \int dy$$

$$u_{xx} = y + \underset{\uparrow}{f(x)}$$

arbitrary function ("constant" with respect to y).

Integrating over x:

$$\int u_{xx}\,dx = \int y\,dx + \int f(x)\,dx$$

$$u_x = xy + \underset{\uparrow}{f_1(x)} + g(y) \leftarrow \text{"constant" with respect to } x$$

integral of an arbitrary function of x is another arbitrary function of x

Integrating over dx again:

$$\int u_x\,dx = \int xy\,dx + \underbrace{\int f_1(x)\,dx}_{\text{arb. function } h(x)} + \int g(y)\,dx$$

$$u = \frac{1}{2}x^2 y + x g(y) + h(x) + \underbrace{f(y)}_{}$$

again, an arbitrary "constant" with respect to x.

So:

$$u = \frac{1}{2}x^2 y + x\,g(y) + f(y) + h(x),$$

where f, g and h are arbitrary functions.

(8) $\quad u_{xx} - u = 0 \qquad (*)$

A similar 2nd-order <u>ordinary</u> differential equation $u'' - u = 0$ for $u(x)$ has the general solution $u = C_1 e^{x} + C_2 e^{-x}$.

Hence, $(*)$ has the solution

$$\underline{u = f(y)\,e^{x} + g(y)\,e^{-x}},$$

where $f(y)$ and $g(y)$ are arbitrary functions of y (i.e. "constants" with respect to x).

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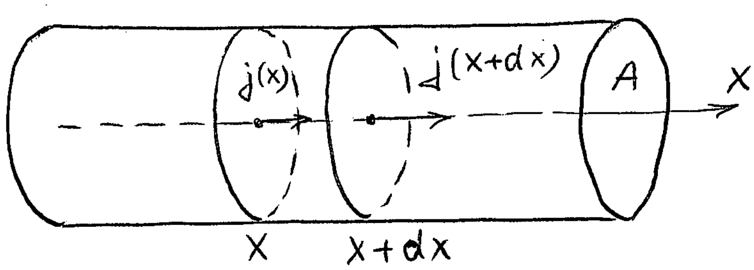

Let q be the heat energy density, i.e. heat energy per unit volume.

The amount of heat between the two planes at x and $x+dx$ is $q\,\underbrace{A\,dx}_{\text{volume of the "slice"}}$.

The amount of heat that flows into the slice in time dt is $j(x)\,A\,dt$, where $j(x)$ is the heat energy flux density at x, and A is the cross section area of the rod.
The amount of heat that flows out of the slice is $j(x+dx)\,A\,dt$ (in time dt).
Since the heat energy is conserved, the change of the heat energy in the slice over the time interval dt can be related to $j(x)$ and $j(x+dx)$, as:

$$\underbrace{\frac{\partial}{\partial t}(q\,A\,dx)\,\cancel{dt}}_{\text{change in the amount of heat energy in the slice}} = j(x)\,A\,\cancel{dt} - j(x+dx)\,A\,\cancel{dt}$$

$$\frac{\partial q}{\partial t}\cancel{A}\,dx = -\left[j(x+dx)-j(x)\right]\cancel{A}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial j}{\partial x}$$

$$\frac{\partial q}{\partial t} + \frac{\partial j}{\partial x} = 0 \qquad (1) \quad \text{energy conservation.}$$

The heat energy density q is proportional to the temperature at x: $q = C\rho T$, where ρ is the density of the rod, and C is the specific heat. The flux density is proportional to the temperature gradient (Fourier's law):

$$j = -\varkappa \frac{\partial T}{\partial x} \qquad (2)$$

where $\varkappa$ is the thermal conductivity. (Minus in this equation is a consequence of heat flowing from places with higher T to those with lower T.)

Substituting (2) and the expression for g into Eq. (1), we have:

$$\frac{\partial}{\partial t}(C\rho T) - \frac{\partial}{\partial x}\left(\varkappa \frac{\partial T}{\partial x}\right) = 0$$

For a uniform rod $C = \text{const}$, $\rho = \text{const}$, $\varkappa = \text{const}$,

hence: $$C\rho \frac{\partial T}{\partial t} - \varkappa \frac{\partial^2 T}{\partial x^2} = 0,$$

or $$\boxed{\frac{\partial u}{\partial t} - K \frac{\partial^2 u}{\partial x^2} = 0}$$

where $K = \dfrac{\varkappa}{C\rho}$, and we denoted the temperature u.

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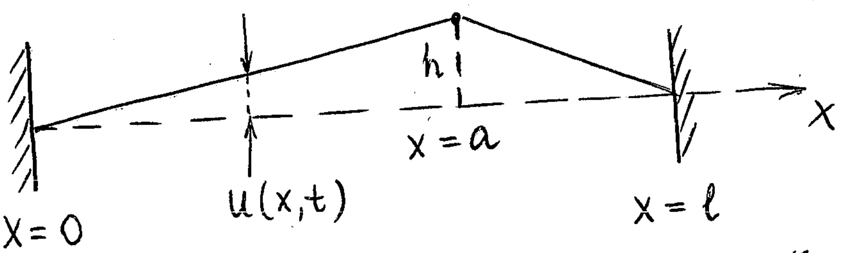

The boundary conditions for the displacement of the string $u(x,t)$ are: $u(0,t) = 0$, $u(l,t) = 0$ (fixed ends).

At $t = 0$ the string has the shape shown on the diagram. This can be written analytically as two linear functions:

$$u(x,0) = \begin{cases} C_1 x, & 0 \le x \le a \\ C_2(l-x), & a \le x \le l \end{cases}$$

C_1 and C_2 are chosen so that $u(a,0) = h$, so that:

$$u(x,0) = \begin{cases} h\dfrac{x}{a}, & 0 \le x \le a \\ h\dfrac{l-x}{l-a}, & a \le x \le l. \end{cases}$$

At $t=0$ the string is at rest, i.e. has zero velocity: $u_t(x,0) = 0$. This and $u(x,0)$ above are the initial conditions.