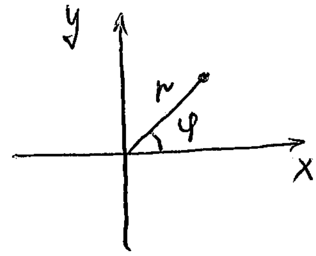
# AMA3006 Problem sheet 2



## Examples

(1) $\nabla^2 u = 0$ in plane polar coordinates $(r, \varphi)$

reads: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0 \quad (1)$$




Seek solution in the form

$$u(r,\varphi) = R(r)\,\Phi(\varphi) \qquad (2)$$

[ main idea of the method of variable separation ]

Substitute (2) into (1):

$$\Phi(\varphi)\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R(r)}{r^2}\,\frac{\partial^2 \Phi}{\partial \varphi^2} = 0 \qquad (3)$$

In fact, partial derivatives here can be replaced with "total" derivatives $\frac{d}{dr}$ and $\frac{d}{d\varphi}$ or $\frac{d^2}{d\varphi^2}$ here, since $R$ and $\Phi$ depend only on $r$ and $\varphi$, respectively.

Dividing (3) by $R(r)\,\Phi(\varphi)$ and replacing $\partial \to d$, we have:

$$\underbrace{\frac{1}{R(r)}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{\text{This function depends only on } r} + \frac{1}{r^2}\underbrace{\frac{1}{\Phi(\varphi)}\frac{d^2\Phi}{d\varphi^2}}_{\text{This expression depends only on } \varphi} = 0$$

This function depends only on $r$

This expression depends only on $\varphi$. For the equation to hold, it must be <u>a constant</u>.

Introducing the separation constant $\lambda$, we have:

$$\frac{1}{\Phi(\varphi)} \Phi''(\varphi) = \lambda \qquad (4)$$

$$\frac{1}{R(r)} \frac{1}{r} (r R')' + \frac{\lambda}{r^2} = 0 \qquad (5)$$

Let us solve (4) first. We will see that $\lambda$ cannot be positive.

1| $\lambda = 0$ (4) $\Rightarrow$ $\Phi''(\varphi) = 0$

$$\Phi(\varphi) = C_1 \varphi + C_2$$

The angular part of the solution must be periodic, i.e. $\Phi(\varphi + 2\pi) = \phi(\varphi)$

Hence, $C_1 = 0$ and $\Phi(\varphi) = C$ (const) (6)

Corresponding radial equation:

$$\frac{1}{R} \frac{1}{r} (r R')' = 0$$

$$(r R')' = 0$$

Integrating:

$$\int \frac{d}{dr} (r R') \, dr = \int 0 \, dr$$

$$r R' = C_1$$

$$\frac{dR}{dr} = \frac{C_1}{r}$$

$$\int \frac{dR}{dr} dr = \int \frac{C_1}{r} dr$$

$$R(r) = C_1 \ln r + C_2 \qquad (7)$$

Combining (6) and (7), we have:

$$u(r,\varphi) = (C_1 \ln r + C_2)\, C$$
$$= \underbrace{C_1 C}_{= C} \ln r + \underbrace{C_2 C}_{= D}$$

Re-naming const.

$$\Rightarrow \quad u(r,\varphi) = C \ln r + D. \qquad (8)$$

2] $\lambda < 0$ : denote $\lambda = -k^2$

(4) $\Rightarrow \quad \Phi'' = -k^2 \Phi$

$$\Phi'' + k^2 \Phi = 0$$

The general solution of this equation is

$$\Phi(\varphi) = A_k \cos k\varphi + B_k \sin k\varphi,$$

where $A_k$ and $B_k$ are arbitrary constants.

By periodicity, $\Phi(\varphi + 2\pi) = \Phi(\varphi)$, we require:

$$A_k \cos(k\varphi + k2\pi) + B_k \sin(k\varphi + k2\pi)$$
$$= A_k \cos k\varphi + B_k \sin k\varphi$$

This holds for all $\varphi$ and arbitrary $A_k$ and $B_k$ only if $k2\pi$ is an integer number of full turns, hence $k$ must be integer (and $\neq 0$, as $\lambda \neq 0$ now).

$\Rightarrow \quad k = 1, 2, \ldots$, i.e., $\underline{\underline{k = n}}$, $\underline{n \in \mathbb{N}}$.

(There is no need to consider negative $k$, since $\cos(-k\varphi) = \cos k\varphi$, $\sin(-k\varphi) = -\sin k\varphi$, and the latter minus sign can be absorbed into $B_k$).

<u>Note:</u> for $\lambda > 0$, e.g. $\lambda = k^2$, we have $\Phi'' - k^2\Phi = 0$, whose solution $\Phi = Ae^{k\varphi} + Be^{-k\varphi}$ CANNOT BE PERIODIC!

Hence: $\Phi_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi$, $n \in \mathbb{N}$.

Equation for $R(r)$:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{n^2}{r^2}R(r)$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2 R(r)$$

Seek solution in the form $R(r) = r^s$

$$R' = s r^{s-1}, \qquad rR' = sr^s$$

$$r(rR')' = r s^2 r^{s-1} = s^2 r^s.$$

Hence:

$$s^2 r^s = n^2 r^s$$

$$s = \pm n,$$

and the solution of the radial equation is a linear combination of the two solutions, with $s = n$ and $s = -n$, and arbitrary coefficients:

$$R_n(r) = E_n r^n + F_n r^{-n}.$$

Hence $\quad u(r,\varphi) = R_n(r)\,\Phi_n(\varphi), \quad n \in \mathbb{N} \qquad (9)$

Using the <u>superposition principle</u>, we can combine (8) and (9) into a more general solution:

with all $n \in \mathbb{N}$

$$\boxed{u(r,\varphi) = C\ln r + D + \sum_{n=1}^{\infty}\left(A_n\cos n\varphi + B_n \sin n\varphi\right) \times \left(E_n r^n + F_n r^{-n}\right)}$$

② 1| $\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx \equiv I_{nm}$

Recall: $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

$$\Rightarrow \quad I_{nm} = \frac{1}{2}\int_0^l \left[\cos\frac{(n-m)\pi x}{l} - \cos\frac{(n+m)\pi x}{l}\right]dx$$

$$= \frac{1}{2}\left[\frac{l}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l}\Big|_0^l - \frac{l}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\Big|_0^l\right]$$

$$= \frac{1}{2}\left[\frac{l}{(n-m)\pi}\sin(n-m)\pi - \frac{l}{(n+m)\pi}\sin(n+m)\pi\right]$$

1) For $n \neq m$ this is zero, since sine functions vanish.

2) For $n = m \neq 0$ the 2nd term is zero and the first one can be found by taking the limit $n-m \to 0$: $\sin(n-m)\pi \simeq (n-m)\pi$, $(n-m)\to 0$

$$\Rightarrow \quad I_{nm} = \frac{l}{2}.$$

$\lim_{x\to 0}\frac{\sin x}{x} = 1$ is the rigorous expression of this.

3) For $n=0$, $m=0$: we can also take the limit in the 2nd term, giving

$$\frac{1}{2}[l - l] = 0.$$

Or, we can simply see from the original integral that it is zero.

$$\Rightarrow \quad \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \begin{cases} 0, & n=m=0 \\ \frac{l}{2}\delta_{nm}, & \text{otherwise} \end{cases}$$

Here $\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$ is the Kronecker delta symbol

21 $\displaystyle\int_{-l}^{l} \sin\frac{n\pi x}{l} \cos\frac{m\pi x}{l}\, dx$

The integrand in this integral is an odd function
( sin is odd, cos is even; their product is odd).
An integral of any odd function $(f(-x) = -f(x))$ over a symmetric interval (e.g., $-l$ to $l$) is equal to zero.

We can also obtain this explicitly, using the identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$$\int_{-l}^{l} \sin\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = \frac{1}{2}\int_{-l}^{l}\left(\sin\frac{(n+m)\pi x}{l} + \sin\frac{(n-m)\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[-\frac{l}{(n+m)\pi}\cos\frac{(n+m)\pi x}{l}\Big|_{-l}^{l} - \frac{l}{(n-m)\pi}\cos\frac{(n-m)\pi x}{l}\Big|_{-l}^{l}\right]$$

$$= \frac{1}{2}\left[-\frac{l}{(n+m)\pi}\left(\cos((n+m)\pi) - \cos(-(n+m)\pi)\right)\right.$$
$$\left. - \frac{l}{(n-m)\pi}\left(\cos((n-m)\pi) - \cos(-(n-m)\pi)\right)\right]$$

cos is an even function, i.e. $\cos(-\theta) = \cos\theta$

$$= \frac{1}{2}\left[-\frac{l}{(n+m)\pi}\left[\cos((n+m)\pi) - \cos((n+m)\pi)\right]\right.$$
$$\left. - \frac{l}{(n-m)\pi}\left[\cos((n-m)\pi) - \cos((n-m)\pi)\right]\right] = 0.$$

Note that the integral is zero when $n = m = 0$, since $\sin\frac{n\pi x}{l}$ and $\sin\frac{m\pi x}{l}$ are both zeros. One can also make a special check for $n = m$ using the double-angle formula ($\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$).

3) $$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx$$

1) $n = m = 0 \Rightarrow \cos\frac{n\pi x}{l} = \cos\frac{m\pi x}{l} = 1$,

and the integral is:

$$\int_0^l 1\,dx = l, \text{ as required.}$$

2) Using $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$,

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = \frac{1}{2}\int_0^l\left[\cos\frac{(n+m)\pi x}{l} + \cos\frac{(n-m)\pi x}{l}\right]dx$$

$$= \frac{1}{2}\left[\frac{l}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\Big|_0^l + \frac{l}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l}\Big|_0^l\right]$$

$$= \frac{l}{2}\left[\frac{\sin(n+m)\pi}{(n+m)\pi} + \frac{\sin(n-m)\pi}{(n-m)\pi}\right]$$

For $n \neq m$ both sin functions give zeros, hence the integral is zero.

The value for $n-m=0$, i.e. $n=m$, can be obtained by taking the limit $n-m \to 0$, and using $\frac{\sin(n-m)\pi}{(n-m)\pi} \xrightarrow[n-m\to 0]{} 1$

Recall: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

At the same time $\sin(n+m)\pi = 0$ and the first term is zero (for $n+m \neq 0$).

Hence for $n = m \neq 0$ the integral is equal to $\frac{l}{2}$.

Therefore:

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = \begin{cases} l, & m=n=0 \\ \frac{l}{2}\delta_{nm}, & \text{otherwise.}\end{cases}$$