AMA3006 Problem sheet 3

Homework problems

SOLUTIONS

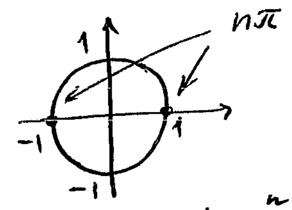
(1) (a) $f(x) = x, \quad -\pi < x < \pi$

This is an odd function, hence $a_n = 0$ and only sine terms contribute to the Fourier series (FS).

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x \sin nx}_{\text{even integrand}}\, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\frac{1}{n}(-\cos nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\, dx \right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}(-\underbrace{\cos n\pi}_{=(-1)^n}) + \frac{1}{n^2}\underbrace{\sin nx\Big|_0^{\pi}}_{=0} \right]$$

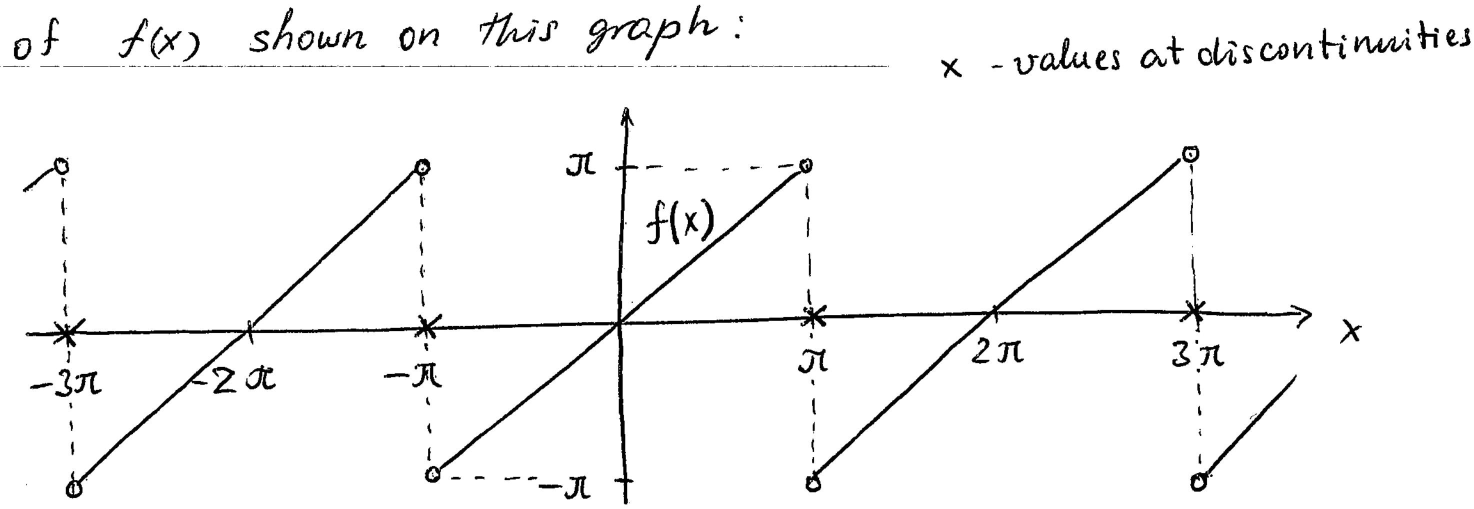


$$= \frac{2}{n}(-1)^{n-1}$$

$\cos n\pi = (-1)^n$
$\sin n\pi = 0$

Hence, $$\boxed{FS = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin nx .}$$

This Fourier series gives a periodic extension of $f(x)$ shown on this graph:



Particular value: $x = \frac{\pi}{2}, \quad f(x) = \frac{\pi}{2}$

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi}{2} \quad \Rightarrow \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$\sin\frac{n\pi}{2} = (-1)^m, \; n = 2m+1$ (odd)
$\phantom{\sin\frac{n\pi}{2}} = 0, \; n = 2m$ (even)

(b) $f(x) = x^2, \quad -\pi \le x \le \pi$.

Here $f(x)$ is even, hence all $b_n = 0$, and the Fourier series contains only cosine terms (and possibly a_0).

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx .$$

$n = 0$: $\quad a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x^2 dx = \dfrac{2}{\pi}\dfrac{x^3}{3} = \dfrac{2\pi^2}{3}$

$n > 0$: $\quad a_n = \dfrac{2}{\pi}\left[\underbrace{x^2 \frac{1}{n}\sin nx \Big|_0^{\pi}}_{0} - \frac{1}{n}\int_0^{x} \sin nx \; 2x \, dx\right]$

(integrating by parts

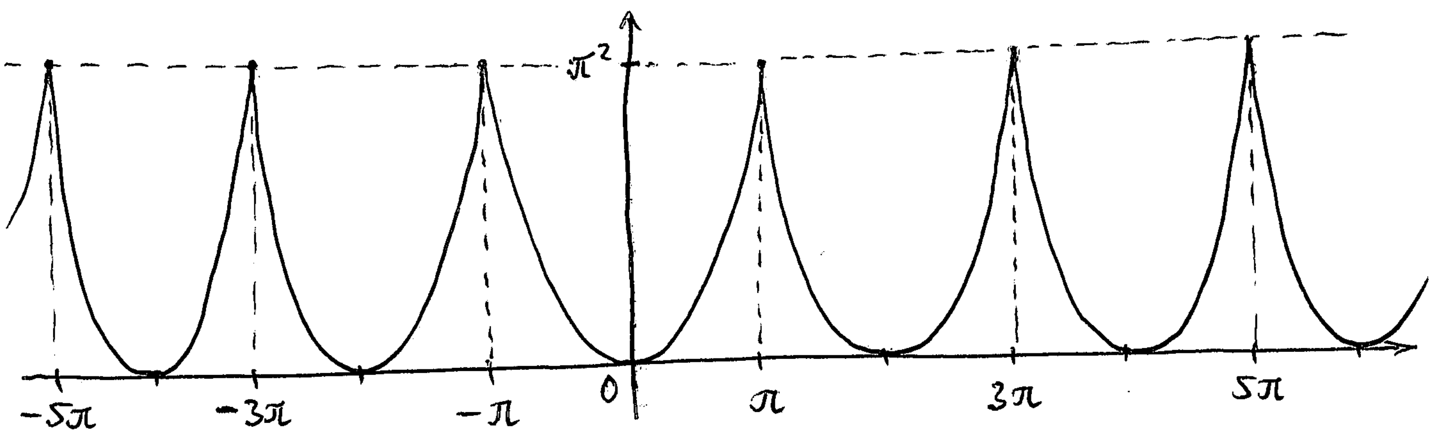
$$= \frac{2}{\pi}\left[-\frac{2}{n^2} x(-\cos nx)\Big|_0^{\pi} - \frac{2}{n^2}\int_0^{\pi} \cos nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\underbrace{\cos n\pi}_{(-1)^n} - \frac{2}{n^3}\underbrace{\sin nx\Big|_0^{\pi}}_{0}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

Hence, the FS is $\underline{\dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}}$.

This Fourier series represents a periodic extension of $f(x) = x^2, \; -\pi \le x \le \pi$:



The extension has no discontinuities.

(c) $f(x) = \sin \alpha x, \quad -\pi < x < \pi, \quad \alpha \notin \mathbb{Z}.$

This is an odd function, so $a_n = 0$, and

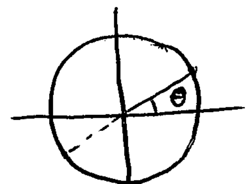
$$b_n = \frac{2}{\pi}\int_0^\pi \sin\alpha x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\left[\cos(\alpha-n)x - \cos(\alpha+n)x\right] dx$$

$$= \frac{1}{\pi}\left[\frac{1}{\alpha-n}\sin(\alpha-n)x - \frac{1}{\alpha+n}\sin(\alpha+n)x\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{\sin(\alpha-n)\pi}{\alpha-n} - \frac{\sin(\alpha+n)\pi}{\alpha+n}\right]$$

$\sin(\alpha\pi + n\pi)$
$= \sin\alpha\pi \, (-1)^n$



$\sin(\theta + n\pi)$
$= (-1)^n \sin\theta$

$$= \frac{1}{\pi}(-1)^n \sin\alpha\pi\left[\frac{1}{\alpha-n} - \frac{1}{\alpha+n}\right]$$

$$= \frac{1}{\pi}(-1)^n \sin\alpha\pi \; \frac{\alpha+n-\alpha+n}{\alpha^2-n^2}$$

$$= \frac{2\sin\alpha\pi}{\pi}(-1)^{n-1}\frac{n}{n^2-\alpha^2}$$

Fourier series: $\quad \sin\alpha x \to \dfrac{2\sin\alpha\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n\sin nx}{n^2-\alpha^2}$.

(2) From Q.1(b)

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}, \quad -\pi \le x \le \pi$$

Using $x = 0$: $\quad 0 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12},$$

i.e. $\quad 1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots = \dfrac{\pi^2}{12}.$

Using $x = \pi$: $\quad \pi^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\dfrac{\cos n\pi}{n^2}$

$(f(-\pi) = f(\pi))$ $\quad \cos n\pi = (-1)^n$

$$\frac{2}{3}\pi^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

i.e. $$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6}.$$

(3) $f(x) = |\sin x|, \quad -\pi \le x \le \pi$

This is an even function, so $b_n = 0$, and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

($\sin x \geq 0$ on $0 \le x \le \pi$, so the modulus sign can be removed).

$n=0$:
$$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{2}{\pi}(-\cos x)\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\Big(-\underbrace{\cos\pi}_{-1} + \underbrace{\cos 0}_{1}\Big) = \frac{4}{\pi}$$

$n>0$:
$$a_n = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{1}{1+n}\left(-\cos(1+n)x\right)\Big|_0^{\pi} + \frac{1}{1-n}\left(-\cos(1-n)x\right)\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{1+n}\Big(1-\underbrace{\cos(n+1)\pi}_{(-1)^{n+1} = (-1)^{n-1}}\Big) + \frac{1}{1-n}\Big(1-\underbrace{\cos(n-1)\pi}_{(-1)^{n-1}}\Big)\right]$$

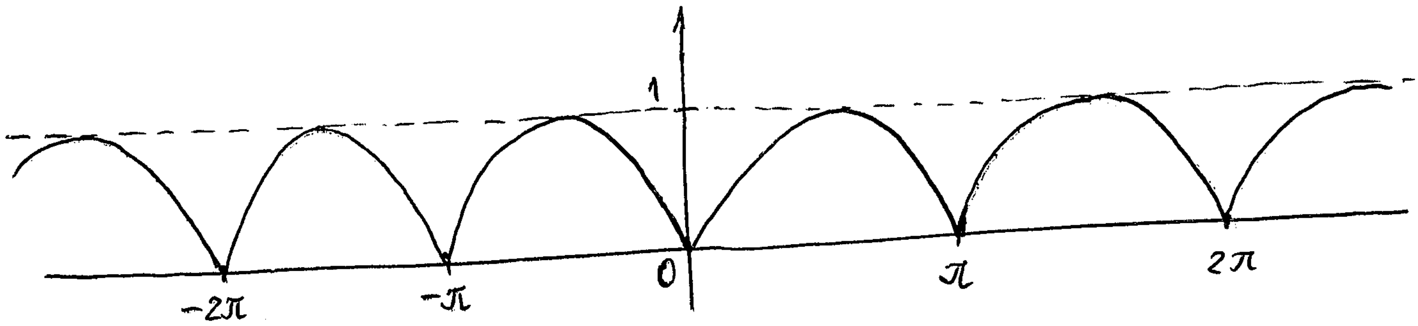
($\cos(1-n)\pi = \cos(n-1)\pi$ even function)

$$= \frac{1}{\pi}\underbrace{\left(1-(-1)^{n-1}\right)}_{\substack{0 \text{ for odd } n\\ 2 \text{ for even } n}}\underbrace{\left[\frac{1}{1+n}+\frac{1}{1-n}\right]}_{\frac{2}{1-n^2}} = \begin{cases} 0, & \text{odd } n \\ -\frac{4}{\pi}\frac{1}{n^2-1}, & n = 2m \text{ (even } n) \\ & m = 1, 2, \ldots \end{cases}$$

Hence, the Fourier series contains only even terms with $n = 2m$, and we have:

$$|\sin x| = \underbrace{\frac{2}{\pi}}_{\frac{a_0}{2}} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2mx}{4m^2-1} \quad , \text{ as required.}$$

This Fourier series describes a periodic extension of $|\sin x|$:



Taking $x = 0$,

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 0}{4m^2-1}$$

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2}{\pi}$$ (we have replaced m by n here)

$$\underline{\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}} \, .$$

[In fact, it is easy to show, e.g., by induction that

$$\sum_{n=1}^{N}\frac{1}{4n^2-1} = \frac{N}{2N+1} \xrightarrow[N\to\infty]{} \frac{1}{2} \, .$$

Or, even simpler, by noticing:

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \dots\right]$$

$$= \frac{1}{2} \, .$$]

(4) $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

The integral is from 0 to π, as $f(x) = 0$ on $-\pi \le x \le 0$

Using some results from Q.3, and noting that these a_n equal $\frac{1}{2}$ of those derived in Q.3, we have

$$a_0 = \frac{2}{\pi}, \qquad a_n = -\frac{2}{\pi}\frac{1}{n^2-1}, \quad n = 2m.$$

$$a_n = 0, \quad n = 2m-1$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

$\underline{n=1}$:

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi}\frac{1}{2}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\Big[\underbrace{x\Big|_0^{\pi}}_{\pi} - \frac{1}{2}\underbrace{\sin 2x\Big|_0^{\pi}}_{=0}\Big] = \frac{1}{2}.$$

$\underline{n \neq 1}$:

$$b_n = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\left[\cos(1-n)x - \cos(1+n)x\right]dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-n}\sin(1-n)x\Big|_0^{\pi} - \frac{1}{1+n}\sin(1+n)x\Big|_0^{\pi}\right]$$

$$= 0.$$

Hence, we have the Fourier series:

$$\underbrace{\frac{1}{\pi} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}}_{\text{This is } \frac{1}{2} \text{ of the Fourier series from Q.3}} + \frac{1}{2}\sin x.$$

This result could have been written straightaway,
by noticing that in this question

$$f(x) = \frac{1}{2}|\sin x| + \frac{1}{2}\sin x$$

Indeed: for $0 \le x \le \pi$ $\sin x \geq 0$, so

this gives $f(x) = \sin x$;

for $-\pi \le x \le 0$ $\sin x \le 0$, so

this gives $f(x) = -\frac{1}{2}\sin x + \frac{1}{2}\sin x = 0$.

* * *

For $x = -\frac{\pi}{2}$ we have: $f(x) = 0$, so that

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2m(-\frac{\pi}{2})}{4m^2-1} + \frac{1}{2}\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{=-1}$$

$$\cos(-m\pi) = (-1)^m$$

$$\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{4m^2-1} = \frac{1}{\pi} - \frac{1}{2}$$

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{4m^2-1} = \frac{1}{2} - \frac{\pi}{4}.$$

[Note: we could also obtain this result by setting $x = \frac{\pi}{2}$, or by setting $x = \frac{\pi}{2}$ in the Fourier series from Q. 3.]

(5) $f(x) = x(\pi - x), \qquad 0 \le x \le \pi$

Expanding this function in a sine Fourier series, we obtain its odd extension.

$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\pi\int_0^\pi x\sin nx\,dx - \int_0^\pi x^2 \sin nx\,dx\right]$$

$$= \underbrace{2\int_0^\pi x\sin nx\,dx}_{\mathrm{I}} - \underbrace{\frac{2}{\pi}\int_0^\pi x^2\sin nx\,dx}_{\mathrm{II}}$$

Integral I: $\displaystyle\int_0^\pi x\sin nx\,dx = x\frac{1}{n}(-\cos nx)\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx$

$$= -\frac{\pi}{n}\underbrace{\cos n\pi}_{(-1)^n} + \frac{1}{n^2}\sin nx\Big|_0^\pi = -(-1)^n\frac{\pi}{n}$$

Integral II: $\displaystyle\int_0^\pi x^2\sin nx\,dx = x^2\frac{1}{n}(-\cos nx)\Big|_0^\pi + \frac{2}{n}\int_0^\pi \cos nx\, x\,dx$

$$= \frac{\pi^2}{n}(-\underbrace{\cos n\pi}_{(-1)^n}) + \underbrace{x\frac{2}{n^2}\sin nx\Big|_0^\pi}_{=0} - \frac{2}{n^2}\int_0^\pi \sin nx\,dx$$

$$= -(-1)^n\frac{\pi^2}{n} + \frac{2}{n^3}\cos nx\Big|_0^\pi$$

$$= -(-1)^n\frac{\pi^2}{n} + \frac{2}{n^3}(\cos n\pi - 1) = -(-1)^n\frac{\pi^2}{n} - \frac{2}{n^3}\left(1-(-1)^n\right)$$

So, we have:

$$b_n = \cancel{-2(-1)^n\frac{\pi}{n}} + \cancel{\frac{2}{\pi}(-1)^n\frac{\pi^2}{n}} + \frac{2}{\pi}\frac{2}{n^3}\left(1-(-1)^n\right)$$

$$= \frac{4}{\pi}\frac{1}{n^3}\left(1-(-1)^n\right) = \begin{cases} 0, & n \text{ even} \\ \dfrac{8}{\pi}\dfrac{1}{n^3}, & n = 2m+1 \text{ (odd)} \end{cases}$$

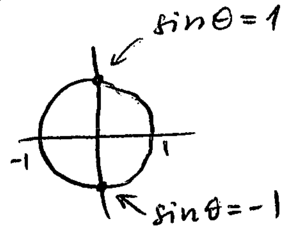
Hence, the sine Fourier series for $f(x)$ is:

$$\frac{8}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3} \qquad (*)$$

For $x = \frac{\pi}{2}$ $\quad x(\pi - x) = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$

In the series $\quad \sin(2m+1)\frac{\pi}{2} = \sin\left(m+\frac{1}{2}\right)\pi$

$$= (-1)^m$$



Hence:

$$\frac{8}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3} = \frac{\pi^2}{4}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3} = \frac{\pi^3}{32}.$$

It is interesting to note that the Fourier series $(*)$,

$$\frac{8}{\pi}\left[\sin x + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \dots\right]$$

has a very large relative contribution of the 1st term. This is a result of the odd-extension of $f(x) = x(\pi - x)$, $0 \le x \le \pi$, being very similar to the $\sin x$ function: