# Fourier's method for PDE.

A piecewise smooth function $f(x)$ defined in the interval $0 \le x \le l$, can be expanded in the Fourier cosine or Fourier sine series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx, \tag{1}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx. \tag{2}$$

## Examples

1. A string of length $l$ fixed at the ends and initially at rest, is struck at its mid-point, so that momentum $P$ is imparted to it. Determine the motion of the string.

   *Hint*: solve the one-dimensional wave equation with the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 0, & 0 < x < \frac{l}{2} - \delta, \\ v_0, & \frac{l}{2} - \delta < x < \frac{l}{2} + \delta, \\ 0, & \frac{l}{2} + \delta < x < l, \end{cases} \tag{3}$$

   where $v_0 = P/(2\delta\rho)$ is the initial velocity over a small interval $2\delta$ near $x = \frac{l}{2}$, and $\rho$ is the linear mass density. Then take the limit $\delta \to 0$.

2. Find the temperature $u(x,t)$ of the rod of length $l$, whose ends are insulated [i.e., no heat flux, $u_x(0,t) = u_x(l,t)$], given the initial temperature distribution $u(x,0) = f(x)$.

3. Determine the vibrations of a string with fixed ends, initially at rest, produced by a load with constant density $q$ applied at $t = 0$ over the whole length of the string.

   [Hint: seek solution in the form $u(x,t) = \tilde{u}(x,t) + u_s(x)$, where $u_s(x)$ is the stationary solution of the wave equation with the load, $u_{tt} - c^2 u_{xx} = q/\rho$.]

## Homework problems

1. Find the temperature of the rod $u(x,t)$, whose ends are kept at zero temperature, given that its initial temperature is $T$, by solving the heat equation

$$u_t - K u_{xx} = 0, \tag{4}$$

   with boundary conditions $u(0,t) = u(l,0) = 0$ and initial condition $u(x,0) = T$.

   *Answer:* $$u(x,t) = \frac{4T}{\pi}\sum_{m=0}^{\infty} \frac{\sin\dfrac{(2m+1)\pi x}{l}}{2m+1} e^{-[(2m+1)^2\pi^2/l^2]Kt}.$$

2. Solve the same problem for the parabolic initial temperature distribution,

$$u(x,0) = \frac{4T}{l^2}x(l-x).$$

   *Answer:* $$u(x,t) = \frac{32T}{\pi^3}\sum_{m=0}^{\infty} \frac{\sin\dfrac{(2m+1)\pi x}{l}}{(2m+1)^3} e^{-[(2m+1)^2\pi^2/l^2]Kt}.$$

3. Solve the heat equation for the rod with insulating ends, $u_x(0,t) = u_x(l,t)$, and initial temperature distribution

$$u(x,0) = \frac{4T}{l^2}x(l-x).$$

*Answer:* $$u(x,t) = \frac{2T}{3} - \frac{16T}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos\dfrac{2m\pi x}{l}}{4m^2}\, e^{-[4m^2\pi^2/l^2]Kt}.$$

[Note that unlike Questions 1 and 2, the temperature here does not tend to zero for $t \to \infty$, since the heat energy cannot escape the rod.]

4. Determine the displacement of the string, $u(x,t)$, with fixed ends, $u(0,t) = u(l,t) = 0$, given that it is initially at rest and has a parabolic shape with maximum displacement $h$,

$$u(x,0) = \frac{4h}{l^2}x(l-x), \quad u_t(x,0) = 0.$$

*Answer:* $$u(x,t) = \frac{32h}{\pi^3}\sum_{m=0}^{\infty}\frac{\sin\dfrac{(2m+1)\pi x}{l}}{(2m+1)^3}\cos\frac{(2m+1)\pi c}{l}t.$$

5. A rectangular membrane of sides $2a$ and $2b$, whose edge is fixed, is acted upon by a uniformly distributed load $q$. The equilibrium shape of the membrane, $u(x,y)$, obeys the equation

$$u_{xx} + u_{yy} = -\frac{q}{T}, \tag{5}$$

where $T$ is the tension per unit length in the membrane, $-a \le x \le a$, $-b \le y \le b$, and

$$u(-a,y) = u(a,y) = u(x,-b) = u(x,b) = 0. \tag{6}$$

   (a) Seek solution of Eq. (5) in the form $u_1(x)$ with boundary conditions $u_1(-a) = 0$, $u_1(a) = 0$, and show that

$$u_1(x) = \frac{q}{2T}(a^2 - x^2). \tag{7}$$

   (b) Let $u(x,y) = \tilde{u}(x,y) + u_1(x)$. By substitution into equation (5) and conditions (6), show that $\tilde{u}(x,y)$ satisfies the homogeneous equation

$$\tilde{u}_{xx} + \tilde{u}_{yy} = 0, \tag{8}$$

   with boundary conditions $\tilde{u}(-a,y) = \tilde{u}(a,y) = 0$, $\tilde{u}(x,-b) = \tilde{u}(x,b) = -u_1(x)$.

   (c) Seek solution of (8) as $\tilde{u}(x,y) = X(x)Y(y)$, with $X(-a) = X(a) = 0$, and show that

$$\tilde{u}(x,y) = A_n \cosh\frac{(n+\frac{1}{2})\pi y}{a}\cos\frac{(n+\frac{1}{2})\pi x}{a}, \tag{9}$$

   where $A_n$ is an arbitrary constant, and $n = 0, 1, 2, \ldots$.

   (d) Using the superposition principle, construct a more general $\tilde{u}(x,y)$ as a sum of (9). By subjecting it to the remaining boundary conditions, $\tilde{u}(x,-b) = \tilde{u}(x,b) = -u_1(x)$, find $A_n$, and show that

$$u(x,y) = \frac{qa^2}{2T}\left[1 - \frac{x^2}{a^2} + \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^3}\frac{\cosh\dfrac{(2n+1)\pi y}{2a}}{\cosh\dfrac{(2n+1)\pi b}{2a}}\cos\frac{(2n+1)\pi x}{2a}\right].$$