Partial Differential Equations AMA3006 — Problem sheet 5

Fourier transforms and their application to PDE.

For a piecewise smooth function $f(x)$ defined on $-\infty < x < \infty$, the (exponential) Fourier transform is

$$F(p) \equiv \mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ipx} dx. \tag{1}$$

Its inverse allows one to find $f(x)$ if $F(p)$ is known:

$$f(x) = \mathcal{F}^{-1}[F] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(p) e^{-ipx} dp. \tag{2}$$

If $f(x)$ has a jump discontinuity at $x = \xi$ then $\mathcal{F}^{-1}[F] = \frac{1}{2}[f(\xi - 0) + f(\xi + 0)]$.

For a function defined on $0 \le x < \infty$ one can use the cosine or sine Fourier transforms:

$$F_c(p) \equiv \mathcal{F}_c[f] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos px \, dx, \qquad f(x) = \mathcal{F}_c^{-1}[F_c] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(p) \cos px \, dp, \tag{3}$$

$$F_s(p) \equiv \mathcal{F}_s[f] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin px \, dx, \qquad f(x) = \mathcal{F}_s^{-1}[F_s] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(p) \sin px \, dp. \tag{4}$$

Examples

1. Consider Laplace's equation $u_{xx} + u_{yy} = 0$ in the half-plane, $-\infty < x < \infty$, $0 \le y < \infty$, for $u(x, y)$ with the boundary condition $u(x, 0) = f(x)$.

   [This problem describes the steady-state temperature distribution in a large (semi-infinite!) room heated by a wall whose temperature is fixed in time but may change along the wall.]

   (a) Consider the Fourier transform $U(p, y) = \mathcal{F}[u]$ with respect to x, and express $\mathcal{F}[u_{yy}]$ and $\mathcal{F}[u_{xx}]$ in terms of $U(p, y)$. Assume that $u(x, y)$, $u_x(x, y) \to 0$ for $x \to \pm\infty$.

   (b) Fourier-transform Laplace's equation with respect to x, and determine $U(p, y)$ using the boundary condition. By performing the inverse Fourier transform, show that

   $$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{y^2 + (\xi - x)^2}. \tag{5}$$

   (c) Using (5), find $u(x, y)$ if $f(x) = T$ for $-a \le x \le a$, and 0 outside this segment. Give this result a geometric interpretation.

2. Consider the heat equation $u_t - u_{xx} = 0$ (where $K = 1$) for a semi-infinite rod, $0 \le x < \infty$, with the initial condition $u(x, 0) = 0$ and boundary condition $u_x(0, t) = -\sigma$ (steady heat flux into the rod). Using the cosine Fourier transform with respect to x, $U_c(p, t) = \mathcal{F}_c[u]$, show that

   $$u(x, t) = \frac{2\sigma}{\pi} \int_0^{\infty} \frac{1 - e^{-p^2 t}}{p^2} \cos px \, dp.$$

Homework problems

1. For $u(x, y)$ defined on $0 \le x < \infty$, with the cosine Fourier transform $U_c(p, y)$ with respect to x, prove that

   $$\mathcal{F}_c[u_{yy}] = \frac{\partial^2 U_c(p, y)}{\partial y^2}, \quad \text{and} \quad \mathcal{F}_c[u_{xx}] = -\sqrt{\frac{2}{\pi}} u_x(0, y) - p^2 U_c(p, y).$$

   [Hint: in the latter, use integration by parts twice.]

2. Consider Laplace's equation $u_{xx} + u_{yy} = 0$ in the quadrant, $0 \le x < \infty$, $0 \le y < \infty$, with the boundary conditions $u_x(0, y) = 0$ and

$$u(x,0) = \begin{cases} T, & 0 \le x \le a, \\ 0, & a < x < \infty. \end{cases}$$

(a) Using the results of Q. 1, perform the cosine Fourier transform of Laplace's equation with respect to x, and take into account the boundary conditions to show that

$$U_c(p, y) = T\sqrt{\frac{2}{\pi}}\frac{\sin pa}{p}e^{-py}. \tag{6}$$

(b) Use the inverse cosine Fourier transform [second equation in (3)], and integrate over p to show that

$$u(x,y) = \frac{T}{\pi}\left[\arctan\frac{a+x}{y} + \arctan\frac{a-x}{y}\right].$$

Hints: $\sin pa \cos px = \frac{1}{2}[\sin(a+x)p + \sin(a-x)p]$, $\int\limits_0^\infty \frac{\sin \alpha p}{p} e^{-\beta p} dp = \arctan\frac{\alpha}{\beta}$.

3. Consider the heat equation $u_t - u_{xx} = 0$ for a semi-infinite rod, $0 \le x < \infty$, with the initial condition $u(x, 0) = 0$ and boundary condition $u(0, t) = T$ (end of the rod has a fixed temperature).

(a) Using the sine Fourier transform with respect to x, $U_s(p, t) = \mathcal{F}_s[u]$, prove that

$$\mathcal{F}_s[u_t] = \frac{\partial U_s(p,t)}{\partial t}, \quad \text{and} \quad \mathcal{F}_s[u_{xx}] = \sqrt{\frac{2}{\pi}}u(0,t)p - p^2U_s(p,t).$$

(b) By applying the sine Fourier transform to the heat equation, show that

$$\frac{\partial U_s(p,t)}{\partial t} + p^2U_s(p,t) - \sqrt{\frac{2}{\pi}}Tp = 0. \tag{7}$$

(c) Solve the differential equation for $U_s(p, t)$ with the appropriate initial condition, and use the inverse sine Fourier transform, second equation in (4), to show that

$$u(x,t) = \frac{2T}{\pi}\int_0^\infty \frac{1-e^{-p^2t}}{p}\sin px\, dp. \tag{8}$$

4. Solve problem 3 for an arbitrary time-dependent boundary condition, $u(0, t) = f(t)$.

<u>Hints:</u> instead of equation (7), you should obtain

$$\frac{\partial U_s(p,t)}{\partial t} + p^2U_s(p,t) - \sqrt{\frac{2}{\pi}}f(t)p = 0.$$

Solve this first-order linear inhomogeneous differential equation by standard methods, to show that

$$U_s(p,t) = \sqrt{\frac{2}{\pi}}pe^{-p^2t}\int_0^t e^{p^2\tau}f(\tau)d\tau.$$

Substitute $U_s(p, t)$ into the inverse sine Fourier transform equation, and integrate over p with the help of $\int_0^\infty e^{-\alpha p^2} p \sin \beta p\, dp = \sqrt{\pi}e^{-\beta^2/4\alpha}\frac{\beta}{4\alpha^{3/2}}$. In the remaining integral over τ, introduce new variable $s = x/(2\sqrt{t-\tau})$, and replace the integration limits accordingly, to obtain

$$u(x,t) = \frac{2}{\sqrt{\pi}}\int_{x/(2\sqrt{t})}^\infty f\left(t - \frac{x^2}{4s^2}\right)e^{-s^2}ds. \tag{9}$$

Show that for $f(t) = T$, Eq. (9) gives $u(x, t) = T\left[1-\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right]$, where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$ is the error function, for which $\lim_{z\to\infty} \mathrm{erf}(z) = 1$. This provides the answer to integral (8).