AMA3006 Problem sheet 6

Examples

(1) (a) $$\mathcal{L}[y''] = \int_0^\infty e^{-pt} y'' dt$$ Integrating by parts

$$= \underbrace{e^{-pt} y' \Big|_0^\infty}_{} - (-p)\int_0^\infty e^{-pt} y' dt$$ And again

This contribution vanishes at $t \to \infty$, so, only lower limit contributes

$$= -y'(0) + pe^{-pt} y\Big|_0^\infty + p^2 \underbrace{\int_0^\infty e^{-pt} y(t) dt}_{Y(p)}$$

$$= \underline{p^2 Y(p) - py(0) - y'(0)}.$$

(b) $$\mathcal{L}[e^{\alpha t}] = \int_0^\infty e^{-pt} e^{\alpha t} dt = \int_0^\infty e^{-(p-\alpha)t} dt$$

$$= -\frac{e^{-(p-\alpha)t}}{p-\alpha}\Big|_0^\infty = \underline{\frac{1}{p-\alpha}}.$$

(c) $$\mathcal{L}[te^{\alpha t}] = \int_0^\infty t e^{\alpha t} e^{-pt} dt$$

Note that $te^{-pt} = -\frac{\partial}{\partial p} e^{-pt}$.

Hence we have

$$\mathcal{L}[te^{\alpha t}] = -\frac{\partial}{\partial p}\int_0^\infty e^{\alpha t} e^{-pt} dt$$

$$= -\frac{\partial}{\partial p}\frac{1}{p-\alpha} = \underline{\frac{1}{(p-\alpha)^2}}.$$

Integration over t and differentiation with respect to p can be interchanged

(d) and (e) : $\mathcal{L}[\cos\omega t]$ and $\mathcal{L}[\sin\omega t]$ can be determined at once using the Euler formula, $e^{i\omega t} = \cos\omega t + i\sin\omega t$.

So, we can take $\mathcal{L}[e^{i\omega t}]$ and determine $\mathcal{L}[\cos\omega t]$ and $\mathcal{L}[\sin\omega t]$ as its real and imaginary parts.

$$\mathcal{L}[e^{i\omega t}] \underset{\uparrow}{=} \frac{1}{p - i\omega} = \frac{p + i\omega}{(p - i\omega)(p + i\omega)}$$

by ① (b)

$$= \frac{p}{p^2 + \omega^2} + i\,\frac{\omega}{p^2 + \omega^2}$$

Hence, $\mathcal{L}[\cos\omega t] = \underline{\dfrac{p}{p^2+\omega^2}}$, $\mathcal{L}[\sin\omega t] = \underline{\dfrac{\omega}{p^2+\omega^2}}$.

(f) $\theta(t - s) = 1$ for $t \geqslant s$

$= 0$ for $t < s$,

so :
$$\mathcal{L}[\theta(t-s)] = \int_0^\infty e^{-pt}\,\theta(t-s)\,dt$$

$$= \int_s^\infty e^{-pt}\,dt = -\frac{e^{-pt}}{p}\bigg|_s^\infty = \underline{\frac{e^{-ps}}{p}}.$$

② (a) Applying the Laplace transform to the differential equation:

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\sin 2t]$$

and using ① (a) and ① (e), we have:

$$p^2 Y(p) + Y(p) = \frac{2}{p^2+4}$$

(where we have used the fact that $y(0) = y'(0) = 0$).

$$\Rightarrow \quad Y(p) = \frac{2}{(p^2+1)(p^2+4)}$$

Using partial fractions, we can write the right-hand side as

$$\frac{2}{(p^2+1)(p^2+4)} = \frac{A}{p^2+1} + \frac{B}{p^2+4}$$

$$A(p^2+4) + B(p^2+1) = 2$$

$$(A+B)p^2 + 4A + B = 2$$

$$\Rightarrow \quad A + B = 0, \quad B = -A, \quad \text{and} \quad 4A - A = 2$$

$$3A = 2 \quad \Rightarrow \quad A = \frac{2}{3}, \quad B = -\frac{2}{3}$$

Hence, $Y(p) = \frac{2}{3}\frac{1}{p^2+1} - \frac{1}{3}\frac{2}{p^2+4}$

$$y(t) = \mathcal{L}^{-1}[Y] = \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{p^2+1}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{2}{p^2+4}\right]$$

$$\underline{y(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t.} \quad \Big\} \text{ Using ① (e)}$$

(b) Applying the Laplace transform to $y''+y=\sin t$ and taking into account $y(0)=y'(0)=0$, we have

$$p^2 Y(p) + Y(p) = \frac{1}{p^2+1}$$

$$Y(p) = \frac{1}{p^2+1} \cdot \frac{1}{p^2+1}$$

The right-hand side cannot be expanded in partial fractions. However, we notice that

$\mathcal{L}^{-1}\left[\frac{1}{p^2+1}\right] = \sin t$, and using the convolution theorem, we obtain

$$y(t) = \underbrace{\int_0^t \sin(t-\tau)\sin\tau \, d\tau}_{\text{convolution of two identical } \sin t \text{ functions.}}$$

$$= \frac{1}{2}\int_0^t \left(\cos(t-2\tau) - \cos t\right) d\tau$$

$$= \frac{1}{2}\int_0^t \left(\cos(2\tau - t) - \cos t\right) d\tau \quad \Big\} \text{ cos is an even function}$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin(2\tau - t)\Big|_0^t - \tau\cos t\Big|_0^t\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(\sin t - \sin(-t)\right) - t\cos t\right]$$

$$\Rightarrow \quad \underline{y(t) = \frac{1}{2}\sin t - \frac{1}{2}t\cos t}\,.$$

③ $u_{tt} - c^2 u_{xx} = 0$ for $u(x,t)$ on $0 \le x < \infty$,

$u(x,0) = u_t(x,0) = 0$, $u_t(0,t) = g(t)$.

We will use the Laplace transform in t:

$U(x,p) = \mathcal{L}[u]$

$$\mathcal{L}[u_{xx}] = \int_0^\infty e^{-pt} \frac{\partial^2 u}{\partial x^2} dt = \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-pt} u(x,t) dt$$

$$= \frac{\partial^2 U(x,p)}{\partial x^2}.$$

$$\mathcal{L}[u_{tt}] = p^2 U(x,p) \underbrace{- p\, u(x,0) - u_t(x,0)}_{\substack{= \\ 0 \\ \text{in this problem}}}$$

(this is similar to question ①(a))

Hence, applying the Laplace transform to the wave equation, we have:

$$p^2 U(x,p) - c^2 \frac{\partial^2 U}{\partial x^2} = 0,$$

or $$\frac{\partial^2 U}{\partial x^2} - \frac{p^2}{c^2} U(x,p) = 0.$$

Solving this differential equation, we have:

$$U(x,p) = A(p) e^{\frac{p}{c}x} + B(p) e^{-\frac{p}{c}x}.$$

$U(x,p)$ (and $u(x,t)$) should not diverge at $x \to \infty$, hence $A(p) = 0$.

$$U(x,p) = B(p) e^{-\frac{p}{c}x}. \qquad (1)$$

To use the boundary condition $u_t(0,t) = g(t)$, take Laplace transform of it: $p\,U(0,p) = G(p)$, where $G(p) = \mathcal{L}[g]$.

Setting $x=0$ in (1) and multiplying it by p, 6

we find:
$$G(p) = p\, B(p)$$
$$\Rightarrow \quad B(p) = \frac{G(p)}{p}$$

Hence: $$U(x,p) = \frac{e^{-\frac{p}{c}x}}{p}\, G(p) \,. \qquad (2)$$

$$G(p) = \mathcal{L}[g], \qquad \frac{e^{-p\frac{x}{c}}}{p} = \mathcal{L}\left[\theta\left(t-\frac{x}{c}\right)\right]$$
(see ① (f)).

The right-hand side is a product of two Laplace transforms. Hence, by the convolution theorem, its original is the convolution of $g(t)$ and $\theta(t-\frac{x}{c})$:

$$u(x,t) = \int_0^t g(\tau)\, \theta\left(t-\frac{x}{c}-\tau\right) d\tau$$

$\theta(t-\frac{x}{c}-\tau) = 1$ for $t-\frac{x}{c}-\tau \geq 0$, hence (and 0 otherwise) only $\tau \leq t-\frac{x}{c}$ contribute.

Hence: $$u(x,t) = \int_0^{t-\frac{x}{c}} g(\tau)\, d\tau \,, \qquad t-\frac{x}{c} \geq 0$$

and 0 otherwise $(t-\frac{x}{c} < 0)$.

or $$u(x,t) = \begin{cases} \int_0^{t-\frac{x}{c}} g(\tau)\, d\tau \,, & x \leq ct \\ 0 \,, & x > ct \end{cases}$$

[The disturbance of the string at the origin $x=0$, reaches only $x \leq ct$ in time t.]