AMA3006 Problem sheet 6

Homework problems

SOLUTIONS

① (a) $$\mathcal{L}[e^{\alpha t} f(t)] = \int_0^\infty e^{-pt} e^{\alpha t} f(t)\,dt$$

$$= \int_0^\infty e^{-(p-\alpha)t} f(t)\,dt$$

$$= \underline{F(p-\alpha)},$$

where $F(p) = \int_0^\infty e^{-pt} f(t)\,dt$.

(b) $$\mathcal{L}[f(t-a)] = \int_0^\infty e^{-pt} f(t-a)\,dt$$

$$= \int_0^\infty e^{-p(t-a)} e^{-pa} f(t-a)\,dt$$

$$= e^{-pa} \int_0^\infty e^{-p(t-a)} f(t-a)\,dt$$

$f(t-a) = 0$ for $t-a<0$, i.e. $t<a$.

Hence, the lower limit can be replaced by a.

Introducing the new variable, $\tau = t-a$, we have:

$$\mathcal{L}[f(t-a)] = e^{-pa} \int_a^\infty e^{-p(t-a)} f(t-a)\,d(t-a)$$

$$= e^{-pa} \int_0^\infty e^{-p\tau} f(\tau)\,d\tau$$

$$= \underline{e^{-pa} F(p)}.$$

② $$\mathcal{L}[y'] = \int_0^\infty e^{-pt}\, y'(t)\, dt$$

$$= \underbrace{e^{-pt}\, y(t)\Big|_0^\infty}_{\text{Here, the upper limit contribution is zero}} - \int_0^\infty (-p)\, e^{-pt}\, y(t)\, dt$$

$$= -y(0) + p \underbrace{\int_0^\infty e^{-pt}\, y(t)\, dt}_{Y(p)}$$

Hence, $\underline{\mathcal{L}[y'] = p\,Y(p) - y(0)}$.

③ $y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = -1$

$\mathcal{L}[y''] = p^2\, Y(p) - p\, y(0) - y'(0)$

$\mathcal{L}[y'] = p\, Y(p) - y(0)$.

Taking the Laplace transform of the differential equation, and making use of the initial conditions, we have:

$$p^2\, Y(p) - p \cdot 1 - (-1) - 3\,(p\, Y(p) - 1) + 2\, Y(p) = 0$$

$$(p^2 - 3p + 2)\, Y(p) = p - 4$$

Factorising the left-hand side, we obtain:

$$Y(p) = \frac{p-4}{(p-1)(p-2)}$$

Using partial fractions, we can expand:

$$\frac{p-4}{(p-1)(p-2)} = \frac{A}{p-2} + \frac{B}{p-1}$$

$$p-4 = A(p-1) + B(p-2)$$

$$p-4 = (A+B)p - A - 2B$$

Hence, $A+B=1 \Rightarrow B = 1-A$

$$-4 = -A - 2B$$

$$A + 2(1-A) = 4$$

$$-A = 2 \Rightarrow A = -2$$

$$B = 3$$

Therefore:

$$Y(p) = -\frac{2}{p-2} + \frac{3}{p-1}$$

$$y(t) = \mathcal{L}^{-1}[Y(p)] = \underline{-2e^{2t} + 3e^{t}}.$$

Here we use $\mathcal{L}^{-1}\left[\frac{1}{p-\alpha}\right] = e^{\alpha t}$

4. $y'' + 2y' = e^{-t}$, $y(0) = y'(0) = 0$.

Applying the Laplace transform and taking into account the initial conditions, we have:

$$p^2 Y(p) + 2p\, Y(p) = \frac{1}{p+1}$$

$\mathcal{L}[e^{\alpha t}] = \frac{1}{p-\alpha}$

$$p(p+2)\, Y(p) = \frac{1}{p+1}$$

$$Y(p) = \frac{1}{p(p+1)(p+2)}$$

Let us expand the right-hand side using partial fractions:

$$\frac{1}{p(p+1)(p+2)} = \frac{A}{p} + \frac{B}{p+1} + \frac{C}{p+2}$$

$$1 = A(p+1)(p+2) + Bp(p+2) + Cp(p+1)$$

$$1 = p^2(A+B+C) + p(3A+2B+C) + 2A$$

$$\begin{cases} A+B+C=0 & (1) \\ 3A+2B+C=0 & (2) \\ 2A=1 & (3) \end{cases}$$

$(3) \Rightarrow \underline{A = \frac{1}{2}}$; $(2)-(1)$ gives: $2A+B=0$

$$B = -2A \Rightarrow \underline{B=-1}$$

$$\underline{C} = -A-B = -\frac{1}{2}+1 = \underline{\frac{1}{2}}.$$

Hence, $$Y(p) = \frac{1}{2p} - \frac{1}{p+1} + \frac{1}{2(p+2)}$$

$$\underline{y(t)} = \mathcal{L}^{-1}[Y(p)] = \underline{\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}}.$$

$\mathcal{L}^{-1}\left[\frac{1}{p}\right] = 1$ for $t \geq 0$ (and 0 for $t<0$)

5 (a) $$y'' + \omega^2 y = f(t).$$

Taking the Laplace transform, we have:

$$p^2 Y(p) - py(0) - y'(0) + \omega^2 Y(p) = F(p),$$

where $F(p) = \mathcal{L}[f]$.

$$(p^2+\omega^2)Y(p) = F(p) + y'(0) + py(0)$$

$$\underline{Y(p) = \frac{F(p) + y'(0) + py(0)}{p^2+\omega^2}}.$$

(b) For $y(0) = y'(0) = 0$

$$Y(p) = \frac{F(p)}{p^2 + \omega^2}$$

This can be written as

$$Y(p) = \underbrace{\frac{1}{\omega} F(p)}_{\mathcal{L}[f]} \underbrace{\frac{\omega}{p^2+\omega^2}}_{\mathcal{L}[\sin \omega t]}$$

The right-hand side is a product of two Laplace transforms.

Using the convolution theorem, we find:

$$y(t) = \frac{1}{\omega} \mathcal{L}^{-1}\left[F(p) \frac{\omega}{p^2+\omega^2} \right]$$

$$y(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin \omega(t-\tau)\, d\tau .$$

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$$y' = 4y - 2z \qquad y(0) = 2,$$
$$z' = 5y + 2z \qquad z(0) = -2.$$

Taking the Laplace transforms of the differential equations, and making use of

$$\mathcal{L}[y'] = p\, Y(p) - y(0)$$
$$\mathcal{L}[z'] = p\, Z(p) - z(0),$$

we have:

$$\begin{cases} p\, Y(p) - 2 = 4\, Y(p) - 2\, Z(p) \\ p\, Z(p) + 2 = 5\, Y(p) + 2\, Z(p) \end{cases}$$

$$\begin{cases} (p-4)\, Y(p) + 2 Z(p) = 2 & (1) \\ -5\, Y(p) + (p-2) Z(p) = -2 & (2) \end{cases}$$

Multiplying (1) by $(p-2)$ and (2) by 2, and subtracting (2) from (1), we have:

$$(p-4)(p-2)\,Y(p) + 10\,Y(p) = 2(p-2) + 4$$

$$(p^2 - 6p + 8 + 10)\,Y(p) = 2p$$

$$Y(p) = \frac{2p}{p^2 - 6p + 18}.$$

From (1):

$$Z(p) = 1 - \frac{1}{2}(p-4)\,Y(p)$$

$$= 1 - \frac{1}{\cancel{2}}(p-4)\frac{\cancel{2}p}{p^2 - 6p + 18}$$

$$= \frac{p^2 - 6p + 18 - (p-4)p}{p^2 - 6p + 18}$$

$$\Rightarrow \quad Z(p) = \frac{-2p + 18}{p^2 - 6p + 18}.$$

Re-writing $Y(p)$ and $Z(p)$:

$$Y(p) = \frac{2p}{p^2 - 6p + 9 + 9} = \frac{2p}{(p-3)^2 + 9} = \frac{2(p-3) + 6}{(p-3)^2 + 9}$$

$$Z(p) = \frac{-2p + 18}{p^2 - 6p + 18} = \frac{-2(p-3) - 6 + 18}{(p-3)^2 + 9} = \frac{-2(p-3) + 12}{(p-3)^2 + 9}$$

$$\Rightarrow \quad Y(p) = \frac{2(p-3)}{(p-3)^2 + 9} + \frac{6}{(p-3)^2 + 9},$$

$$Z(p) = -\frac{2(p-3)}{(p-3)^2 + 9} + \frac{12}{(p-3)^2 + 9}.$$

Using $\mathcal{L}^{-1}\left[\frac{\omega}{p^2+\omega^2}\right] = \sin\omega t$

$\mathcal{L}^{-1}\left[\frac{p}{p^2+\omega^2}\right] = \cos\omega t$

together with the 1st shift theorem,

$$\mathcal{L}^{-1}[F(p-\alpha)] = e^{\alpha t} f(t),$$

we obtain:

$$\underline{y(t)} = \mathcal{L}^{-1}[Y(p)] = \underline{2e^{3t}\cos 3t + 2e^{3t}\sin 3t},$$

$$\underline{z(t)} = \mathcal{L}^{-1}[Z(p)] = \underline{-2e^{3t}\cos 3t + 4e^{3t}\sin 3t}.$$

7 $\quad u_{tt} - c^2 u_{xx} = 0, \qquad 0 \le x < \infty$

$u(x,0) = 0, \quad u_t(x,0) = 0$ - initial conditions

$u(0,t) = f(t)$ - boundary condition.

$$\mathcal{L}[u_{tt}] = p^2 U(x,p) \underbrace{- p\,u(x,0) - u_t(x,0)}_{=0 \text{ in this problem}}$$

$$\mathcal{L}[u_{xx}] = \int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-pt}u(x,t)\,dt$$

$$= \frac{\partial^2 U(x,p)}{\partial x^2}.$$

Applying the Laplace transform to the wave equation, we have:

$$p^2 U(x,p) - c^2\frac{\partial^2 U}{\partial x^2} = 0,$$

or
$$\frac{\partial^2 U}{\partial x^2} - \frac{p^2}{c^2}U = 0.$$

This is an ordinary differential equation in x for $U(x,p)$, and its general solution is: (8

$$U(x,p) = A(p)\, e^{\frac{p}{c}x} + B(p)\, e^{-\frac{p}{c}x}$$

arbitrary functions of p.

Since our solution should not diverge for $x \to \infty$, we must set $A(p)=0$, so that

$$U(x,p) = B(p)\, e^{-\frac{p}{c}x}.$$

Using the initial condition $u(0,t) = f(t)$, and taking its Laplace transform,

$$U(0,p) = F(p), \quad \text{where} \quad F(p) = \mathcal{L}[f],$$

we see that $B(p) = F(p)$.

Hence: $\underline{U(x,p) = F(p)\, e^{-\frac{p}{c}x}}$.

$$u(x,t) = \mathcal{L}^{-1}[U(x,p)] = \mathcal{L}^{-1}\left[F(p)\, e^{-\frac{x}{c}p}\right]$$

By the 2nd shift theorem, we have:

$$u(x,t) = f\left(t - \frac{x}{c}\right)$$

This answer is valid for $t - \frac{x}{c} \geq 0$, i.e. $x \leq ct$. For $t - \frac{x}{c} < 0$, $f(t-\frac{x}{c}) = 0$.

So, finally:

$$u(x,t) = \begin{cases} f\left(t - \frac{x}{c}\right), & x \leq ct \\ 0, & x > ct \end{cases}.$$

This solution shows that the disturbance of the string at $x=0$ propagates along the string with velocity c.