Partial Differential Equations AMA3006 — Problem sheet 7

Orthogonal systems of functions.

The *inner product* of two functions, f and g, on the segment $[a, b]$, is defined as

$$(f, g) = \int_a^b f(x)g(x)dx, \tag{1}$$

and the *norm* of f is $\|f\| = \sqrt{(f, f)}$. Two functions are orthogonal if their inner product is zero. A system of functions, $\varphi_1(x)$, $\varphi_2(x)$, ..., is *orthonormal*, if

$$(\varphi_i, \varphi_j) = \delta_{ij}. \tag{2}$$

Given a system of linearly independent functions, $v_1(x)$, $v_2(x)$, ..., an orthonormal system can be constructed by the Gram-Schmidt orthogonalisation process:

$$\begin{aligned}
&\varphi_1 = v_1/\|v_1\|, \\
&\varphi_2 = \tilde{v}_2/\|\tilde{v}_2\|, \quad \text{where} \quad \tilde{v}_2 = v_2 - (\varphi_1, v_2)\varphi_1, \\
&\varphi_3 = \tilde{v}_3/\|\tilde{v}_3\|, \quad \text{where} \quad \tilde{v}_3 = v_3 - (\varphi_1, v_3)\varphi_1 - (\varphi_2, v_3)\varphi_2, \quad \text{etc.}
\end{aligned}$$

An expansion in the orthonormal system of functions, $s_n(x) = \sum_{i=1}^n c_i\varphi_i(x)$, gives the best approximation *in the mean* for a function $f(x)$, when the norm of the difference between them, $\|f - s_n\|$, takes the smallest possible value. This occurs for

$$c_i = (\varphi_i, f). \tag{3}$$

If the system of functions φ_i is *complete*, $\lim_{n\to\infty} \|f - s_n\| = 0$ for any piecewise continuous f.

The notions of orthogonality and norm can be generalised by defining the inner product as

$$(f, g) = \int_a^b f(x)g(x)\rho(x)dx, \tag{4}$$

where $\rho(x) \geq 0$ is the *weight* function. In this case, if $(f, g) = 0$, we say that 'f and g are orthogonal with the weight function ρ.'

Examples

1. Using the Gram-Schmidt process, construct the first three functions of an orthonormal system on $[-1, 1]$ from $v_0 = 1$, $v_1 = x$, $v_2 = x^2$, ...

   Answer: $\varphi_0(x) = \frac{1}{\sqrt{2}}$, $\varphi_1(x) = \sqrt{\frac{3}{2}}\,x$, $\varphi_2(x) = \sqrt{\frac{5}{2}}\,\frac{3x^2-1}{2}$.

   Continuing this process, one obtains a system of orthonormal functions on $[-1, 1]$, $\varphi_n(x) = \sqrt{\frac{2n+1}{2}}P_n(x)$, where $P_n(x)$ are the *Legendre* polynomials ($n = 0, 1, \ldots$).

2. Using the Gram-Schmidt process, construct the first three functions of an orthonormal system on $(-\infty, \infty)$ from $v_0 = 1$, $v_1 = x$, $v_2 = x^2$, ..., with the weight function $\rho = e^{-x^2}$. [Useful integral $\int_{-\infty}^{\infty} x^{2n}e^{-x^2}dx = \sqrt{\pi}(2n-1)!!/2^n$, where $(2n-1)!! = 1 \cdot 3 \cdot \ \ldots \ \cdot (2n-1)$.]

   Answer: $\varphi_0(x) = 1/\pi^{1/4}$, $\varphi_1(x) = 2x/\sqrt{2\sqrt{\pi}}$, $\varphi_2(x) = (4x^2 - 2)/\sqrt{8\sqrt{\pi}}$.

   Continuing this process, one obtains a system of functions orthonormal on $(-\infty, \infty)$ with weight e^{-x^2}, $\varphi_n(x) = H_n(x)/\sqrt{2^n n!\sqrt{\pi}}$, where $H_n(x)$ are the *Hermite* polynomials,

   $$H_n(x) = (-1)^n e^{x^2}\frac{d^n e^{-x^2}}{dx^n}.$$

   Use integration by parts to prove $\int_{-\infty}^{\infty} x^m e^{-x^2}H_n(x)dx = 0$ for $m < n$, hence $(H_m, H_n) = 0$.

Homework problems

1. Prove that the infinite system of functions,

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \quad \frac{\cos 2x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \ldots,$$

is orthonormal on the interval $[0, 2\pi]$ (with the weight function $\rho = 1$).

[Verify that these functions are mutually orthogonal, and their norms are equal to unity.]

2. It can be shown that the Legendre polynomials $P_n(x)$ from example 1 are given by Rodrigues' formula, as the nth derivative of $(x^2 - 1)^n$,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}. \tag{5}$$

[Note: $(x^2 - 1)^n$ is a polynomial of degree $2n$, and its nth derivative is a polynomial of degree n. The coefficient in equation (5) is chosen so that $P_n(1) = 1$.]

   (a) Using Rodrigues' formula, verify the expressions for $\varphi_1(x)$ and $\varphi_2(x)$ obtained in example 1, and show that $\varphi_3(x) = \sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)$.

   (b) The function $(x^2-1)^n$ and its first $n-1$ derivatives, $[(x^2-1)^n]^{(k)}$ $(k = 1, 2, \ldots, n-1)$, vanish at $x = 1$ and -1. Using this property, and repeatedly integrating by parts, show that $P_n(x)$ is orthogonal to x^m for $m < n$, i.e.,

$$\int_{-1}^{1} x^m P_n(x) dx = \frac{1}{2^n n!} \int_{-1}^{1} x^m [(x^2 - 1)^n]^{(n)} dx = 0.$$

   (c) Hence, prove the orthogonality of the Legendre polynomials for $m < n$,

$$\int_{-1}^{1} P_m(x) P_n(x) dx = 0.$$

3. (a) Show that the *Chebyshev* polynomials, $T_n(x) = \cos(n \arccos x)$ $(n = 0, 1, 2, \ldots)$ are mutually orthogonal on $-1 \le x \le 1$ with the weight function $\rho(x) = 1/\sqrt{1 - x^2}$, and determine their norm $\|T_n\|$.
   [Hint: use variable substitution $x = \cos\theta$, i.e., $\theta = \arccos x$.]

   (b) Show that $T_n(\cos\theta)$ is indeed a polynomial in $\cos\theta$ of degree n.
   [Hints: use $\cos(n\theta) = \operatorname{Re} e^{in\theta}$, together with the Euler formula for $e^{i\theta}$, and binomial expansion.]

4. (a) Using the Gram-Schmidt process, construct the first three functions of an orthonormal system on $[0, \infty)$ from $v_0 = 1$, $v_1 = x$, $v_2 = x^2$, $\ldots$, with the weight function $\rho = e^{-x}$.
   [Make use of the integral $\int_0^\infty x^n e^{-x} dx = n!$.]
   Answer: $\varphi_0(x) = 1$, $\varphi_1(x) = x - 1$, $\varphi_2(x) = \frac{1}{2}(x^2 - 4x + 2)$.

   (b) Continuing this process, one obtains a system of functions orthonormal on $[0, \infty)$ with weight e^{-x}, $\varphi_n(x) = (-1)^n L_n(x)/n!$, where $L_n(x)$ are the *Laguerre* polynomials,

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}). \tag{6}$$

   Using repeated integration by parts, show that for $m < n$,

$$(x^m, L_n) = \int_0^\infty x^m e^{-x} L_n(x) dx = 0, \quad \text{and} \quad \|L_n\|^2 = \int_0^\infty e^{-x} L_n^2(x) dx = (n!)^2.$$

   [The first relation in fact proves that $(L_m, L_n) = 0$ for $m < n$. To derive the second relation, present one L_n using (6), and use $L_n(x) = (-1)^n x^n + \ldots$ for the other.]