AMA3006 Problem sheet 7

Examples

① $v_0 = 1, \quad v_1 = x, \quad v_2 = x^2, \quad \dots$; weight function $\rho = 1$.

1) $$\|v_0\|^2 = \int_{-1}^{1} v_0^2 \, dx = \int_{-1}^{1} dx = 2,$$

hence $$\underline{\varphi_0(x)} = \frac{v_0}{\|v_0\|} = \underline{\frac{1}{\sqrt{2}}}.$$

2) $$\tilde{v}_1 = v_1 + c\,\varphi_0$$

c must be chosen to make $\tilde{v}_1$ orthogonal to φ_0:

$$(\varphi_0, \tilde{v}_1) = (\varphi_0, v_1) + c\,\underbrace{(\varphi_0, \varphi_0)}_{\|\varphi_0\|^2 = 1} = 0$$

$$\Longrightarrow \quad c = -(\varphi_0, v_1)$$

$$\tilde{v}_1 = v_1 - (\varphi_0, v_1)\,\varphi_0$$

$$(\varphi_0, v_1) = \int_{-1}^{1} \frac{1}{\sqrt{2}}\, x \, dx = \frac{1}{\sqrt{2}} \left.\frac{x^2}{2}\right|_{-1}^{1} = 0$$

[This is a result of integrating an odd function over a symmetric interval.]

$$\Rightarrow \quad \tilde{v}_1 = x$$

$$\|\tilde{v}_1\|^2 = \int_{-1}^{1} \tilde{v}_1^2 \, dx = \int_{-1}^{1} x^2 dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}$$

$$\Rightarrow \quad \|\tilde{v}_1\| = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad \underline{\varphi_1(x)} = \frac{\tilde{v}_1}{\|\tilde{v}_1\|} = \underline{\sqrt{\frac{3}{2}}\, x}.$$

3) $\tilde{v}_2 = v_2 + c_0 \varphi_0 + c_1 \varphi_1$,

where c_0 and c_1 are chosen to make $\tilde{v}_2$ orthogonal to φ_0 and φ_1:

$$(\varphi_0, \tilde{v}_2) = (\varphi_0, v_2) + c_0 \underbrace{(\varphi_0, \varphi_0)}_{=1} + c_1 \underbrace{(\varphi_0, \varphi_1)}_{=0} = 0$$

$$\Rightarrow \quad c_0 = -(\varphi_0, v_2).$$

$$(\varphi_1, \tilde{v}_2) = (\varphi_1, v_2) + c_0 \underbrace{(\varphi_1, \varphi_0)}_{=0} + c_1 \underbrace{(\varphi_1, \varphi_1)}_{=1} = 0$$

$$\Rightarrow \quad c_1 = -(\varphi_1, v_2).$$

$$c_0 = -(\varphi_0, v_2) = -\int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 dx = -\frac{1}{\sqrt{2}} \frac{x^3}{3}\Big|_{-}^{1} = -\frac{\sqrt{2}}{3}$$

$$c_1 = -(\varphi_1, v_2) = -\int_{-1}^{1} \sqrt{\frac{3}{2}}\, x\, x^2 dx = 0 \quad \Big\} \text{ odd integrand, symmetric interval}$$

$$\Rightarrow \quad \tilde{v}_2 = x^2 - \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}}$$

$$\tilde{v}_2 = x^2 - \frac{1}{3}$$

$$\|\tilde{v}_2\|^2 = \int_{-1}^{1} v_2^2 dx = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= \frac{x^5}{5}\Big|_{-1}^{1} - \frac{2}{3}\frac{x^3}{3}\Big|_{-1}^{1} + \frac{1}{9}x\Big|_{-1}^{1}$$

$$= \frac{2}{5} - \frac{2}{3}\frac{2}{3} + \frac{2}{9} = \frac{18-20+10}{45} = \frac{8}{45}$$

$$\|\tilde{v}_2\| = \sqrt{\frac{8}{45}} = \frac{2}{3}\sqrt{\frac{2}{5}}$$

$$\varphi_2(x) = \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right) = \sqrt{\frac{5}{2}}\ \frac{3x^2-1}{2}.$$

(2) $v_0 = 1, \quad v_1 = x, \quad v_2 = x^2, \dots;$ *weight function* $\rho = e^{-x^2}$.

1) $$\|v_0\|^2 = \int_{-\infty}^{+\infty} 1 \cdot e^{-x^2} dx = \sqrt{\pi}$$

[Here and below we make use of the integral

$$\int_{-1}^{1} x^{2n} e^{-x^2} dx = \frac{\sqrt{\pi}\,(2n-1)!!}{2^n},$$

where $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$

and by convention $(-1)!! = 1$.]

$$\varphi_0(x) = \frac{v_0}{\|v_0\|} = \frac{1}{\pi^{1/4}}.$$

2) Proceeding as in (1):

$$(\varphi_0, v_1) = \int_{-\infty}^{+\infty} \frac{1}{\pi^{1/4}} x e^{-x^2} dx = 0$$ } odd integrand, symmetric interval

$$\Rightarrow \tilde{v}_1 = v_1 - (\varphi_0, v_1)\varphi_0 = v_1 = x.$$

$$\|\tilde{v}_1\|^2 = \int_{-\infty}^{+\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}\cdot 1}{2} = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \varphi_1(x) = \frac{\tilde{v}_1}{\|\tilde{v}_1\|} = \frac{x}{\sqrt{\frac{\sqrt{\pi}}{2}}} = \frac{\sqrt{2}\,x}{\sqrt{\sqrt{\pi}}} = \frac{2x}{\sqrt{2\sqrt{\pi}}}.$$

3) Similarly to 1 3),

$$(\varphi_0, v_2) = \int_{-\infty}^{+\infty} \frac{1}{\pi^{1/4}} x^2 e^{-x^2} dx = \frac{1}{\pi^{1/4}} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi^{1/4}}{2}$$

$$(\varphi_1, v_2) = \int_{-\infty}^{+\infty} \frac{2x}{\sqrt{2\sqrt{\pi}}} x^2 e^{-x^2} dx = 0$$ } odd integrand, symmetric interval

$$\Rightarrow \quad \tilde{v}_2 = x^2 - \frac{\pi^{1/4}}{2} \cdot \frac{1}{\pi^{1/4}} = x^2 - \frac{1}{2}.$$

$$\|\tilde{v}_2\|^2 = \int_{-\infty}^{+\infty} \left(x^2 - \frac{1}{2}\right)^2 e^{-x^2} dx$$

$$= \int_{-\infty}^{+\infty} \left(x^4 - x^2 + \frac{1}{4}\right) e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}\, 3}{4} - \frac{\sqrt{\pi}}{2} + \frac{1}{4}\sqrt{\pi} = \frac{\sqrt{\pi}\,(3 - 2 + 1)}{4} = \frac{\sqrt{\pi}}{2}$$

$$\varphi_2(x) = \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \frac{x^2 - \frac{1}{2}}{\sqrt{\frac{\sqrt{\pi}}{2}}} = \frac{\sqrt{2}\left(x^2 - \frac{1}{2}\right)}{\sqrt{\sqrt{\pi}}} = \frac{2x^2 - 1}{\sqrt{2\sqrt{\pi}}},$$

or

$$\underline{\varphi_2(x) = \frac{4x^2 - 2}{\sqrt{8\sqrt{\pi}}}}.$$

$H_0(x) = 1$, $H_1(x) = 2x$ and $H_2(x) = 4x^2 - 2$ are the first 3 Hermite polynomials.

In general,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}$$

and

$$\varphi_n(x) = \frac{H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}.$$

For example, for $n=2$:

$$(-1)^2 e^{x^2}\left(e^{-x^2}\right)'' = e^{x^2}\left(-2x\, e^{-x^2}\right)'$$

$$= e^{x^2}\left(-2e^{-x^2} + (-2x)^2 e^{-x^2}\right)$$

$$= 4x^2 - 2, \text{ as we obtained above.}$$

Let us prove that $H_n(x)$ is orthogonal to x^m for $m<n$ (with weight function e^{-x^2}).

$$(x^m, H_n) = \int_{-\infty}^{+\infty} x^m H_n(x)\, e^{-x^2} dx = \int_{-\infty}^{+\infty} x^m (-1)^n \cancel{e^{x^2}} \frac{d^n e^{-x^2}}{dx^n} \cancel{e^{-x^2}} dx$$

$$= (-1)^n \int_{-\infty}^{+\infty} x^m \frac{d^n e^{-x^2}}{dx^n} dx$$

Integrating by parts

$$= \underbrace{(-1)^n x^m \frac{d^{n-1} e^{-x^2}}{dx^{n-1}} \Bigg|_{-\infty}^{+\infty}}_{=0} - (-1)^n m \int_{-\infty}^{+\infty} x^{m-1} \frac{d^{n-1} e^{-x^2}}{dx^{n-1}} dx$$

$= 0$, because e^{-x^2} is always present in the derivatives, and $e^{-x^2} \xrightarrow[x\to\pm\infty]{} 0$

Again, by parts

$$= \underbrace{(-1)^{n-1} m\, x^{m-1} \frac{d^{n-2} e^{-x^2}}{dx^{n-2}}}_{=0} - (-1)^{n-1} m(m-1) \int_{-\infty}^{+\infty} x^{m-2} \frac{d^{n-2} e^{-x^2}}{dx^{n-2}} dx$$

After m integrations by parts, we have:

$$(-1)^{n-m} m! \int_{-\infty}^{+\infty} \frac{d^{n-m} e^{-x^2}}{dx^{n-m}} dx = (-1)^{n-m} m! \frac{d^{n-m-1} e^{-x^2}}{dx^{n-m-1}} \Bigg|_{-\infty}^{+\infty} = \underline{\underline{0}}.$$

($n-m>0$ is important!)

Hence, $(x^m, H_n) = 0$.

Since $H_m(x)$ is a polynomial of degree m, $\sum_{k=0}^{m} a_k x^k$,

$$(H_m, H_n) = \left(\sum_{k=0}^{m} a_k x^k, H_n\right) = \sum_{k=0}^{m} a_k \underbrace{(x^k, H_n)}_{k<n \text{ here}} = \underline{\underline{0}},$$

i.e. H_m and H_n are orthogonal.