Partial Differential Equations AMA3006 **Problem sheet 8**

Sturm-Liouville problem.

Any second-order linear differential expression, $\tilde{p}u'' + \tilde{r}u' - \tilde{q}u$, can be transformed into the *self-adjoint form*, $L[u] = (pu')' - qu$, by multiplying it by

$$R(x) = e^{\int [(\tilde{r}-\tilde{p}')/\tilde{p}]dx}. \tag{1}$$

Green's formula for the self-adjoint operator L:

$$\int_a^b (vL[u] - uL[v])dx = p(u'v - v'u)\big|_a^b . \tag{2}$$

Sturm-Liouville eigenvalue problem: determine the values of λ, for which the equation,

$$(pu')' - qu + \lambda\rho u = 0, \tag{3}$$

with $p(x) > 0$ and $\rho(x) \geq 0$ on $[a, b]$, has nontrivial solutions u which satisfy one of the following homogeneous boundary conditions:

1. $u(a) = u(b) = 0,$
2. $u'(a) = u'(b) = 0,$
3. $\alpha_a u(a) + \beta_a u'(a) = 0, \quad \alpha_b u(b) + \beta_b u'(b) = 0,$
4. $u(a) = u(b), \quad p(a)u'(a) = p(b)u'(b) \quad [p(a) = p(b)$ gives periodic boundary conditions].

If $p(a) = 0$ (or $p(b) = 0$) the Sturm-Liouville problem is *singular*, and we require that u is bounded at $x = a$ (or $x = b$), or its growth is restricted.

Examples

1. Transform the following equations into the self-adjoint form, $L[u] + \lambda\rho u = 0$:

$$(1 - x^2)u'' - 2xu' + \lambda u = 0, \tag{4}$$

$$xu'' + (1 - x)u' + \lambda u = 0, \tag{5}$$

   and find the corresponding weight functions ρ.

2. (a) Seek the solution of (4) in the form $u(x) = \sum_{k=0}^{\infty} a_k x^k$, and show that

$$a_{k+2} = -\frac{\lambda - k(k+1)}{(k+1)(k+2)}\, a_k. \tag{6}$$

   (b) Hence, show that the solution is a polynomial if $\lambda = n(n + 1)$, where $n = 0, 1, \ldots.$

   (c) Prove that $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n(x^2-1)^n}{dx^n}$ satisfies equation (4) with $\lambda = n(n + 1)$.

   [Consider $v = (x^2 - 1)^n$, show that $(x^2 - 1)v' = 2nxv$, and differentiate it $(n + 1)$ times.]

3. (a) Seek the solution of (5) in the form $u(x) = \sum_{k=0}^{\infty} a_k x^k$, and show that

$$a_{k+1} = -\frac{\lambda - k}{(k+1)^2}\, a_k. \tag{7}$$

   (b) Hence, show that the solution is a polynomial if $\lambda = n$, where $n = 0, 1, \ldots.$

Note: Legendre polynomials $P_n(x)$ solve the Sturm-Liouville problem (4) on $[-1, 1]$, and the polynomial solutions of (5) on $[0, \infty)$ orthogonal with e^{-x}, are the Laguerre polynomials $L_n(x)$.

Homework problems

1. Find the function $R(x)$ which makes the differential operator

$$x(1-x)\frac{d^2}{dx^2}+[q-(p+1)x]\frac{d}{dx}$$

self-adjoint. *Answer:* $R(x)=x^{q-1}(1-x)^{p-q}$.

[Solutions of the Sturm-Liouville problem $x(1-x)u''+[q-(p+1)x]u'+\lambda u=0$ on $[0,1]$ are the *Jacobi* polynomials; they are orthogonal with the weight function $x^{q-1}(1-x)^{p-q}$.]

2. (a) Transform the differential equation

$$u''-2xu'+\lambda u=0. \tag{8}$$

into the self-adjoint form and show that $\rho=e^{-x^2}$.

(b) Seek the solution of (8) in the form $u(x)=\sum_{k=0}^{\infty}a_kx^k$, and show that

$$a_{k+2}=-\frac{\lambda-2k}{(k+1)(k+2)}\,a_k. \tag{9}$$

Hence, show that the solution is a polynomial if $\lambda=2n$, where $n=0,\,1,\ldots$.

These solutions of the Sturm-Liouville problem on $(-\infty,\infty)$, are the Hermite polynomials; they are orthogonal with the weight function e^{-x^2}.

3. (a) Show that the equation

$$(1-x^2)u''-xu'+\lambda u=0,$$

written in the self-adjoint form, takes the form

$$\left(\sqrt{1-x^2}\,u'\right)'+\frac{\lambda}{\sqrt{1-x^2}}\,u=0. \tag{10}$$

(b) Consider the Sturm-Liouville problem for equation (10) on $-1\le x\le 1$, with the boundary conditions that $u(x)$ is finite at $x=\pm1$. Show that $u=T_n(x)=\cos(n\arccos x)$ is an eigenfunction of this problem, and find the corresponding eigenvalue λ.

Note: By the Sturm-Liouville theory, the Chebyshev polynomials, $T_n(x)$, are orthogonal with the weight function $(1-x^2)^{-1/2}$, as we have already seen in Problem sheet 7.

4. Find all the eigenvalues and normalised eigenfunctions of the Sturm-Liouville problem:

$$y''+\lambda y=0,\quad 0\le x\le\pi/2,\quad y(0)=0,\quad y'(\pi/2)=0.$$

Answer: $\lambda=(2n+1)^2$, $y_n=\frac{2}{\sqrt{\pi}}\sin(2n+1)x$, $n=0,\,1,\ldots$.

[Hints: consider three cases, $\lambda=k^2>0$, $\lambda=0$, and $\lambda=-k^2$; to normalise a solution, e.g., $y=A\sin kx$, choose A so that $\int_0^{\pi/2}y^2dx=1$.]

5. Find all the eigenvalues and normalised eigenfunctions of the Sturm-Liouville problem:

$$y''+\lambda y=0,\quad 0\le x\le\pi,\quad y(0)+y'(0)=0,\quad y(\pi)+y'(\pi)=0.$$

Answer: $\lambda=-1$, $y_{-1}=\sqrt{\frac{2}{1-e^{-2\pi}}}\,e^{-x}$; $\lambda=n^2$, $y_n=\sqrt{\frac{2}{\pi(n^2+1)}}\,(n\cos nx-\sin nx)$, $n=1,\,2,\ldots$.

[Hint: again, consider three cases, $\lambda=k^2>0$, $\lambda=0$, and $\lambda=-k^2$.]