AMA3006 Problem sheet 8

Homework problems

SOLUTIONS

(1) $$x(1-x)\frac{d^2}{dx^2} + [q-(p+1)x]\frac{d}{dx}$$

To make this operator self-adjoint, multiply by $R(x)$:

$$R(x)\,x(1-x)\frac{d^2}{dx^2} + R(x)[q-(p+1)x]\frac{d}{dx}$$

and require

$$\left(R\,x(1-x)\right)' = R[q-(p+1)x]$$

$$R'x(1-x) + R(1-2x) = R[q-(p+1)x]$$

$$R'x(1-x) = R[q-(p+1)x-1+2x]$$

$$R'x(1-x) = R[q-1-(p-1)x]$$

$$\frac{dR}{dx} = R\,\frac{q-1-(p-1)x}{x(1-x)}$$

$$\int\frac{dR}{R} = \int\frac{q-1-(p-1)x}{x(1-x)}\,dx$$

For the integral on the right-hand side, use partial fractions:

$$\frac{q-1-(p-1)x}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

$$q-1-(p-1)x = A(1-x)+Bx$$

$$q-1-(p-1)x = A-(A-B)x$$

$\underline{A = q-1}$, $\quad A-B = p-1 \Rightarrow \underline{B} = -p+1+A = \underline{q-p}$

$$\ln R = \int \frac{q-1}{x}\,dx + \int \frac{q-p}{1-x}\,dx$$

$$\ln R = (q-1)\ln x - (q-p)\ln(1-x) + C$$

choose $C=0$

$$\ln R = (q-1)\ln x + (p-q)\ln(1-x)$$

$$\ln R = \ln x^{q-1} + \ln (1-x)^{p-q}$$

$$\underline{R(x) = x^{q-1}(1-x)^{p-q}}.$$

Note: the differential operator, written in the self-adjoint form, is:

$$R\,x(1-x)\frac{d^2}{dx^2} + R[q-(p+1)x]\frac{d}{dx}$$

$$= \frac{d}{dx}\left(x^{q}(1-x)^{p-q+1}\frac{d}{dx}\right)$$

② (a) $u'' - 2xu' + \lambda u = 0$

To transform this equation into the self-adjoint form, multiply by R

$$Ru'' - R2xu' + \lambda Ru = 0$$

and require

$$R' = -2xR$$

$$\frac{dR}{R} = -2x\,dx$$

$$\ln R = -x^2 + C \qquad (\text{choose } C=0)$$

$$\underline{R = e^{-x^2}}$$

The equation now is: $(e^{-x^2}u')' + \lambda \underbrace{e^{-x^2}}_{\rho}u = 0$.

ρ weight function

(b) Seek solution of $u'' - 2xu' + \lambda u = 0$

in the form $u = \sum_{k=0}^{\infty} a_k x^k$

$$u' = \sum_{k=0}^{\infty} a_k k x^{k-1}, \quad u'' = \sum_{k=0}^{\infty} a_k k(k-1) x^{k-2}.$$

Substituting into the equation:

$$\underbrace{\sum_{k=0}^{\infty} a_k k(k-1) x^{k-2}}_{\substack{\text{In this sum only} \\ \text{terms with } k \geq 2 \\ \text{contribute. Changing}}} - \sum_{k=0}^{\infty} 2a_k k x^k + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

$k-2 \to k$

$k-1 \to k+1$

$k \to k+2$

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{k=0}^{\infty} 2a_k k x^k + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

$$\sum_{k=0}^{\infty} \big[\underbrace{a_{k+2}(k+2)(k+1) - 2k a_k + \lambda a_k}_{= 0} \big] x^k = 0$$

$$a_{k+2}(k+2)(k+1) = -(\lambda - 2k) a_k$$

$$\boxed{a_{k+2} = -\frac{\lambda - 2k}{(k+2)(k+1)} a_k}$$

For the solution to be a polynomial, the series should truncate. If we choose $\lambda = 2n$ ($n = 0, 1, 2, \ldots$) then $a_{n+2} = -\frac{2n - 2n}{(n+2)(n+1)} a_n = 0$, so that a_n is the large nonzero coefficient, and the solution is a polynomial of degree n.

[Note: if $\lambda \neq n$ then for $k \gg \lambda$ $a_{k+2} \approx \frac{2k}{(k+2)(k+1)} a_k \approx \frac{2}{k+2} a_k$ and the solution is an infinite series which can be shown to behave as $\sim e^{x^2}$ for large x.]

③ (a) $$(1-x^2)u'' - x u' + \lambda u = 0 \qquad (1)$$

$$R(x)(1-x^2)u'' - R(x)\, x\, u' + \lambda R(x) u = 0$$

For the self-adjoint form, we require:

$$(R(1-x^2))' = -Rx$$

$$R'(1-x^2) - 2xR = -Rx$$

$$R'(1-x^2) = Rx$$

$$\frac{dR}{dx} = R\frac{x}{1-x^2}$$

$$\int \frac{dR}{R} = \int \frac{x\,dx}{1-x^2} \qquad \left\{ \int \frac{x\,dx}{1-x^2} = -\frac{1}{2}\int\frac{d(1-x^2)}{1-x^2} = -\frac{1}{2}\ln(1-x^2) + C \right.$$

$$\ln R = -\frac{1}{2}\ln(1-x^2) + \underbrace{C}_{=0 \text{ (choice is ours)}}$$

$$\ln R = -\ln\sqrt{1-x^2}$$

$$R = \frac{1}{\sqrt{1-x^2}}$$

Hence, equation (1) written in the self-adjoint form reads:

$$\left(\sqrt{1-x^2}\,u'\right)' + \frac{\lambda}{\sqrt{1-x^2}}u = 0 .$$

(b) $$u = T_n(x) = \cos(n\arccos x)$$

Let us work out the derivatives first.

$$u' = -\sin(n\arccos x)\cdot n\left(-\frac{1}{\sqrt{1-x^2}}\right) \qquad \Big\} \text{ Using chain rule}$$

$$u'' = -\cos(n\arccos x)\, n^2\frac{1}{1-x^2} - \sin(n\arccos x)\, n\frac{(-2x)}{2(1-x^2)^{3/2}} \qquad \Big\} \text{ Chain rule, product rule.}$$

Substituting into equation (1), we have:

$$(1-x^2)\left[-\cos(n\arccos x)\frac{n^2}{1-x^2}+\sin(n\arccos x)\frac{nx}{(1-x^2)^{3/2}}\right]$$

$$-x\sin(n\arccos x)\frac{n}{\sqrt{1-x^2}}+\lambda\cos(n\arccos x)=0$$

$$-\cos(n\arccos x)\,n^2+\cancel{\sin(n\arccos x)\frac{nx}{\sqrt{1-x^2}}}$$

$$-\cancel{\sin(n\arccos x)\frac{nx}{\sqrt{1-x^2}}}+\lambda\cos(n\arccos x)=0$$

$$(\lambda-n^2)\cos(n\arccos x)=0.$$

This is true for all x if $\underline{\lambda=n^2}$.

Hence, $\lambda=n^2$ is an eigenvalue of the given Sturm-Liouville problem, and $u=T_n(x)=\cos(n\arccos x)$ is the corresponding eigenfunction.

4)

$$y''+\lambda y=0, \quad 0\le x\le \frac{\pi}{2}$$

$$y(0)=0, \quad y'\left(\frac{\pi}{2}\right)=0.$$

We need to find <u>all</u> values of λ for which the differential equation with the boundary conditions given has nontrivial (i.e. nonzero) solutions.

1) Let $\lambda>0$, $\lambda=k^2$ (i.e, $k=\sqrt{\lambda}$).

$y''+k^2y=0$ has the general solution

$$y=A\cos kx+B\sin kx$$

$y(0)=0$ gives $A\underbrace{\cos 0}_{1} + B\underbrace{\sin 0}_{0} = 0$

$\Rightarrow \quad A=0$, and $y = B\sin kx$.

$y' = Bk\cos kx$, and using $y'(\frac{\pi}{2})=0$,

we have: $Bk\cos(k\frac{\pi}{2}) = 0$.

Hence, $\frac{\pi}{2}k = \frac{\pi}{2} + n\pi$, $n = 0,1,2,\dots$ $(k>0)$.

$k = 2n+1$

$\underline{\lambda = (2n+1)^2}$,

and the corresponding solution is

$$y_n = B\sin(2n+1)x .$$

Let us normalise it:

$$\int_0^{\pi/2} y^2 dx = B^2\int_0^{\pi/2} \sin^2(2n+1)x\, dx = 1.$$

$$\int_0^{\pi/2} \sin^2(2n+1)x\,dx = \int_0^{\pi/2} \frac{1}{2}\left[1-\cos(2(2n+1)x)\right] dx$$

$$= \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2}\cos(2(2n+1)x)\,dx$$

$$= \frac{\pi}{4} - \frac{1}{4(2n+1)}\sin(4n+2)x\Big|_0^{\pi/2}$$

$$= \frac{\pi}{4} - \frac{1}{4(2n+1)}\underbrace{\sin(2n+1)\pi}_{0} = \frac{\pi}{4}.$$

Hence, $B^2\frac{\pi}{4} = 1 \Rightarrow B = \frac{2}{\sqrt{\pi}}$,

and the <u>normalised</u> eigenfunction is $\underline{y_n = \frac{2}{\sqrt{\pi}}\sin(2n+1)x}$.

2) Can $\lambda = 0$ also be an eigenvalue?

$y'' = 0$ gives $y = Ax + B$

$y(0) = 0$ gives $B = 0$, so that $y = Ax$.

$y' = A$ must satisfy $y'(\frac{\pi}{2}) = 0$, hence $A = 0$,
and we do not have nontrivial solutions here.

3) How about $\lambda < 0$? Let $\lambda = -k^2$.

$y'' - k^2 y = 0$ has the general
solution $y = C_1 e^{kx} + C_2 e^{-kx}$,
which can also be written as

$$y = A\cosh kx + B\sinh kx$$

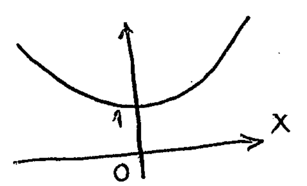
($\cosh kx$ and $\sinh kx$ are linear combinations
of e^{kx} and e^{-kx}, $\cosh kx = \frac{e^{kx}+e^{-kx}}{2}$, $\sinh kx = \frac{e^{kx}-e^{-kx}}{2}$).

Using $y(0) = 0$, we obtain $A = 0$, hence

$$y = B\sinh kx$$

$$y' = Bk\cosh kx$$

Using $y'(\frac{\pi}{2}) = 0$: $Bk\cosh\frac{\pi}{2}k = 0$.

This equation <u>does not</u> have solutions for
$k > 0$, (Recall the graph of $\cosh x$: )
except $B = 0$. So, we do not
have any nontrivial solutions for $\lambda < 0$.

So, <u>finally</u>, the Sturm-Liouville problem has
the eigenvalues $\lambda = (2n+1)^2$ $(n = 0, 1, \ldots)$,
and the corresponding normalised eigenfunctions are

$$y_n = \frac{2}{\sqrt{\pi}}\sin(2n+1)x.$$

5 $y'' + \lambda y = 0, \quad 0 \le x \le \pi$

$y(0) + y'(0) = 0, \quad y(\pi) + y'(\pi) = 0.$

1) Let $\lambda = k^2 > 0$, (and we assume $k > 0$).

$y'' + k^2 y = 0$

has the general solution

$y = A\cos kx + B\sin kx$

$y' = -Ak\sin kx + Bk\cos kx$

Using these in the two boundary conditions:

$y(0) + y'(0) = 0$ gives $A + Bk = 0 \Rightarrow \underline{A = -Bk}$

2nd boundary condition gives:

$-Bk\cos\pi k + B\sin\pi k + Bk^2\sin\pi k + Bk\cos\pi k = 0$

$B \quad (1+k^2)\sin\pi k = 0$

For a nontrivial solution we must have $B \neq 0$,

so that $\sin\pi k = 0$

$\pi k = \pi n, \quad n = 1, 2, \dots$

$k = n$

$\underline{\lambda = n^2}$: $\underline{\text{the eigenvalue}}$.

Then $y = -Bn\cos nx + B\sin nx$

or $y = C(n\cos nx - \sin nx)$ } Here we introduced $C = -B$.

To <u>normalise</u> the eigenfunction

we require $\int_0^\pi y^2\,dx = 1$,

$$C^2 \int_0^\pi (n\cos nx - \sin nx)^2\,dx = 1.$$

$$\int_0^\pi (n\cos nx - \sin nx)^2 dx$$

$$= \int_0^\pi (n^2\cos^2 nx - 2n\cos nx \sin nx + \sin^2 nx)\, dx$$

$$= n^2 \int_0^\pi \frac{1}{2}(1+\cos 2nx)\, dx - n\int_0^\pi \sin 2nx\, dx$$

$$+ \int_0^\pi \frac{1}{2}(1-\cos 2nx)\, dx$$

} Rearranging the integrals

$$= \frac{1}{2}(n^2+1)\underbrace{\int_0^\pi dx}_{\pi} + \frac{1}{2}(n^2-1)\underbrace{\int_0^\pi \cos 2nx\, dx}_{\frac{1}{2n}\sin 2nx\Big|_0^\pi = 0} - n\underbrace{\int_0^\pi \sin 2nx\, dx}_{-\frac{1}{2n}\cos 2nx\Big|_0^\pi = 0}$$

$$= \frac{\pi}{2}(n^2+1).$$

Hence, $C^2\,\frac{\pi}{2}(n^2+1) = 1 \Rightarrow C = \sqrt{\frac{2}{\pi(n^2+1)}}$,

and the normalised eigenfunction is

$$y_n = \sqrt{\frac{2}{\pi(n^2+1)}}\,(n\cos nx - \sin x).$$

2) $\lambda = 0$

$$y'' = 0 \quad\Rightarrow\quad y = Ax + B, \quad y' = A$$

Using the boundary conditions

$$B + A = 0, \qquad A\pi + B + A = 0$$

$$B = -A \quad\Rightarrow\quad A\pi - A + A = 0$$

$$A\pi = 0$$

$$A = 0$$

So, we do not find nontrivial solution here, and $\lambda = 0$ is not an eigenvalue.

3) $\lambda = -k^2 < 0$ ($k > 0$ is assumed)

$y'' - k^2 y = 0$

has the general solution $y = Ae^{kx} + Be^{-kx}$.

$y' = Ake^{kx} - Bke^{-kx}$. Using these in the boundary conditions, we have:

$x=0$: $A + B + Ak - Bk = 0 \Rightarrow A(1+k) + B(1-k) = 0$

$x=\pi$: $Ae^{\pi k} + Be^{-\pi k} + Ake^{\pi k} - Bke^{-\pi k} = 0$.

$$A(1+k)e^{\pi k} + B(1-k)e^{-\pi k} = 0$$

$$A = -B\frac{1-k}{1+k} \quad (\text{from } x=0 \text{ condition})$$

so,

$$-B(1-k)e^{\pi k} + B(1-k)e^{-\pi k} = 0$$

$$-B(1-k)(e^{\pi k} - e^{-\pi k}) = 0.$$

$e^{\pi k} > 1$ $\quad$ $e^{-\pi k} < 0$

this bracket cannot be zero.

The only solution of this equation is $\underline{k = 1,}$

i.e. $\lambda = -1$ is an $\underline{\text{eigenvalue.}}$

Then $A = 0$ and the eigenfunction is

$$y = Be^{-x}.$$

Normalising it: $B^2\int_0^\pi (e^{-x})^2 dx = 1$

$$B^2\int_0^\pi e^{-2x} dx = 1$$

$$B^2\left(-\frac{1}{2}e^{-2x}\right)\Big|_0^\pi = 1$$

$$\frac{B^2}{2}(1 - e^{-2\pi}) = 1 \rightarrow B = \sqrt{\frac{2}{1-e^{2\pi}}}.$$

Summarising, the Sturm - Liouville problem has one negative eigenvalue,

$\lambda = -1$ with normalised eigenfunction

$$y_{-1} = \sqrt{\frac{2}{1-e^{-2\pi}}}\, e^{-x},$$

and infinitely many positive eigenvalues

$$\lambda = n, \qquad n = 1, 2, \dots$$

with normalised eigenvalues

$$y_n = \sqrt{\frac{2}{\pi(n^2+1)}}\,(n\cos nx - \sin x).$$

<u>Note:</u> in the lectures we proved that the eigenvalues of the Sturm - Liouville problem, $(pu')' - qu + \lambda\rho u = 0$, for $q \geq 0$ are non-negative, for all 4 types of the boundary conditions, including type 3,

$$\alpha_a y(a) + \beta_a y'(a) = 0$$
$$\alpha_b y(b) + \beta_b y'(b) = 0$$

with the additional requirement that

$$\frac{\alpha_a}{\beta_a} \leq 0, \qquad \frac{\alpha_b}{\beta_b} \geq 0.$$

In the problem solved above, this requirement is violated by the 1st boundary condition $y(0) + y'(0) = 0$ as $\alpha_0 = 1$, $\beta_0 = 1$ and $\frac{\alpha_0}{\beta_0} > 0$. Because of that our problem admitted <u>one negative eigenvalue</u>.