Partial Differential Equations AMA3006 — Problem sheet 9

Normal forms of 2nd-order linear and quasi-linear PDE.

A second-order linear or quasi-linear[1] equation in two variables, x and y,

$$au_{xx} + 2bu_{xy} + cu_{yy} + g(x, y, u, u_x, u_y) = 0, \tag{1}$$

where a, b and c are functions or x and y, can be transformed to new independent variables,

$$\xi = \phi(x, y), \quad \eta = \psi(x, y), \tag{2}$$

using the following expressions for the derivatives,

$$u_x = u_\xi \phi_x + u_\eta \psi_x, \qquad u_y = u_\xi \phi_y + u_\eta \psi_y, \tag{3}$$
$$u_{xx} = u_{\xi\xi}\phi_x^2 + 2u_{\xi\eta}\phi_x\psi_x + u_{\eta\eta}\psi_x^2 + u_\xi\phi_{xx} + u_\eta\psi_{xx} \tag{4}$$
$$u_{xy} = u_{\xi\xi}\phi_x\phi_y + u_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + u_{\eta\eta}\psi_x\psi_y + u_\xi\phi_{xy} + u_\eta\psi_{xy} \tag{5}$$
$$u_{yy} = u_{\xi\xi}\phi_y^2 + 2u_{\xi\eta}\phi_y\psi_y + u_{\eta\eta}\psi_y^2 + u_\xi\phi_{yy} + u_\eta\psi_{yy}. \tag{6}$$

Equation (1) thus assumes the form

$$\alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} + \tilde{g}(\xi, \eta, u, u_\xi, u_\eta) = 0, \tag{7}$$

where the new coefficients are

$$\alpha = a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2, \tag{8}$$
$$\beta = a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y, \tag{9}$$
$$\gamma = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2. \tag{10}$$

They obey the following relation (verify!)

$$\beta^2 - \alpha\gamma = (b^2 - ac)(\phi_x\psi_y - \phi_y\psi_x)^2. \tag{11}$$

Consider an auxiliary quadratic equation,

$$a\lambda^2 + 2b\lambda + c = 0, \tag{12}$$

with discriminant $\Delta = b^2 - ac$ and roots $\lambda_{1,2} = \frac{1}{a}(-b \pm \sqrt{\Delta})$. Depending on Δ, the PDE is:

- *Hyperbolic*, $\Delta > 0$. For the normal form we require $\alpha = \gamma = 0$; ξ and η are found by solving

  $$\phi_x - \lambda_1\phi_y = 0, \qquad \psi_x - \lambda_2\psi_y = 0. \tag{13}$$

  and the PDE is written in the normal form as $u_{\xi\eta} + \cdots = 0$.[2]

- *Parabolic*, $\Delta = 0$. For the normal form we require $\alpha = \beta = 0$; the variable ξ is found from

  $$a\phi_x + b\phi_y = 0,$$

  and η is arbitrary, such that $\gamma \neq 0$ (e.g., $\eta = x$). The normal form reads $u_{\eta\eta} + \cdots = 0$.

- *Elliptic*, $\Delta < 0$. Equation (12) has complex conjugate roots. Solving the first of equations (13) we find a complex ξ, and set $\eta = \xi^*$. Regarding these variables as independent, we obtain the PDE in the complex normal form $u_{\xi\eta} + \cdots = 0$.

  Introducing real variables, $\rho = \frac{\xi + \eta}{2}$ and $\sigma = \frac{\xi - \eta}{2i}$, we obtain $4u_{\xi\eta} = u_{\rho\rho} + u_{\sigma\sigma}$, and arrive at the elliptic PDE in the real normal form, $u_{\rho\rho} + u_{\sigma\sigma} + \cdots = 0$.

Examples of the hyperbolic, parabolic and elliptic PDE are, respectively, the wave equation, $u_{tt} - c^2u_{xx} = 0$, the heat equation, $u_t - Ku_{xx} = 0$, and the Laplace equation, $u_{xx} + u_{yy} = 0$.

[1] *Quasi-linear* means that the equation is linear with respect to the 2nd-order derivatives.

[2] To solve equation of the form $\frac{\partial\phi}{\partial x} + p(x, y)\frac{\partial\phi}{\partial y} = 0$, write the solution of $\frac{dy}{dx} = p(x, y)$ as $\phi(x, y) = \text{const}$.

Examples

1. Determine the types of the equations and reduce them to normal forms:

$$2u_{xx} + 3u_{xy} + u_{yy} + 7u_x + 4u_y - 2u = 0,$$
$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} - 2yu_x + ye^{y/x} = 0.$$

2. Transform the equation into the normal form in each domain where its type is preserved:

$$u_{xx} + xu_{yy} = 0.$$

3. Reduce to the normal form and solve subject to the boundary conditions:

$$y(x - y^2)(4y^2 u_{xx} - u_{yy}) = 8y^3 u_x + (5y^2 - x)u_y, \qquad u(0, y) = -y^2,\ u_x(0, y) = 1.$$

Homework problems

1. Determine the types of the following equations and reduce them to normal forms:

   (a) $u_{xx} - 2u_{xy} + u_{yy} + 9u_x + 9u_y - 9u = 0$
   Answer: parabolic, $u_{\eta\eta} + 18u_\xi + 9u_\eta - 9u = 0$, $\xi = x + y$, $\eta = x$.

   (b) $u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 15u_y + 27x = 0$
   Answer: hyperbolic, $u_{\xi\eta} - 2u_\xi + u_\eta + \xi + \eta = 0$, $\xi = x + y$, $\eta = 2x - y$.

   (c) $u_{xx} + 2u_{xy} + 5u_{yy} - 32u = 0$
   Answer: elliptic, complex $u_{\xi\eta} - 2u = 0$, $\xi = y - (1 - 2i)x$, $\eta = y - (1 + 2i)x$; real $u_{\rho\rho} + u_{\sigma\sigma} - 8u = 0$, $\rho = y - x$, $\sigma = 2x$.

2. Transform the equation into the normal form in each domain where its type is preserved:

$$u_{xx} + yu_{yy} = 0.$$

   Answers:
   $y < 0$, hyperbolic, $\xi = x + 2\sqrt{-y}$, $\eta = x - 2\sqrt{-y}$, $u_{xx} + yu_{yy} = 4u_{\xi\eta} + \frac{2}{\xi-\eta}(u_\xi - u_\eta) = 0$;
   $y > 0$, elliptic, $\xi = x + 2i\sqrt{y}$, $\eta = x - 2i\sqrt{-y}$, and $\rho = x$, $\sigma = 2\sqrt{y}$, $u_{\rho\rho} + u_{\sigma\sigma} - \frac{1}{\sigma}u_\sigma = 0$.

3. Show that $u_{xx} + 2xu_{xy} + x^2 u_{yy} = 0$ is parabolic, and reduces to the normal form:

$$4(\eta - \xi)u_{\eta\eta} + u_\eta - u_\xi = 0,$$

   where ξ is chosen appropriately and $\eta = y + x^2/2$.

4. Reduce $u_{xx} - u_{yy} = 16xy$ to normal form. [Answer: $u_{\xi\eta} = \eta^2 - \xi^2$, $\xi = y + x$, $\eta = y - x$.] Show that its general solution is $u = \frac{1}{3}\xi\eta(\eta^2 - \xi^2) + g(\xi) + h(\eta)$, with arbitrary g and h. Find the solution $u(x, y)$ which satisfies $u(0, y) = 2y^2$, $u_x(0, y) = 0$. [Answer: $u = \frac{8}{3}x^3 y + 2y^2 + 2x^2$.]

5. Find the general solution of the equation $u_{xx} + 4u_{yy} = 8xy$. Find also the particular solution for which $u(0, y) = y^2$, $u_x(0, y) = 0$. [Answer: $u = \frac{4}{3}x^3 y + y^2 - 4x^2$.]

6. Reduce the equation

$$y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$$

   to the normal form, and hence solve it.

7. Determine the type of the PDE and reduce it to the normal form:

$$u_{xx} + 2\sin x u_{xy} - (\cos^2 x - \sin^2 x)u_{yy} + \cos x u_y = 0.$$

   [Answer: PDE is hyperbolic for $x \neq \pi(n + \frac{1}{2})$, $n \in \mathbb{Z}$, $u_{\xi\eta} + \frac{\xi-\eta}{2[4-(\xi-\eta)^2]}(u_\xi - u_\eta) = 0$, where $\xi = y + \cos x + \sin x$, $\eta = y + \cos x - \sin x$.]