AMA 3006 Problem sheet 9

Examples

(1) 1)
$$2u_{xx}+3u_{xy}+u_{yy}+7u_x+4u_y-2u=0$$

$$a=2,\quad b=\frac{3}{2},\quad c=1$$

$$\Delta=b^2-ac=\frac{9}{4}-2=\frac{1}{4}>0 \Rightarrow \text{PDE is } \underline{\text{hyperbolic}}.$$

Auxiliary equation $2\lambda^2+2\cdot\frac{3}{2}\lambda+1=0$

has roots:
$$\lambda_{1,2}=\frac{1}{2}\left(-\frac{3}{2}\pm\sqrt{\frac{1}{4}}\right)=\frac{1}{2}\left(-\frac{3}{2}\pm\frac{1}{2}\right)$$

$$\lambda_1=-\frac{1}{2},\qquad \lambda_2=-1.$$

To find new variables, ξ and η, we solve:

$$\phi_x+\frac{1}{2}\phi_y=0,$$
or $\frac{dy}{dx}=\frac{1}{2}$
$$\int 2\,dy=\int dx$$
$$2y=x+C$$
$$2y-x=C$$
$$\Rightarrow \underline{\xi=2y-x}$$

$$\psi_x+1\,\phi_y=0,$$
or $\frac{dy}{dx}=1.$
$$\int dy=\int dx$$
$$y=x+C$$
$$y-x=C$$
$$\Rightarrow \underline{\eta=y-x}$$

Expressing the derivatives in new variables:

$$u_x=u_\xi(-1)+u_\eta(-1),\qquad u_y=u_\xi 2+u_\eta$$

$$u_{xx}=u_{\xi\xi}(-1)^2+2u_{\xi\eta}(-1)^2+u_{\eta\eta}(-1)^2=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$$

$$u_{xy}=u_{\xi\xi}(-2)+u_{\xi\eta}((-1)\cdot 2+(-1)1)+u_{\eta\eta}(-1)=-2u_{\xi\xi}-3u_{\xi\eta}-u_{\eta\eta}$$

$$u_{yy}=u_{\xi\xi}\,4+2u_{\xi\eta}\,2+u_{\eta\eta}\cdot 1=4u_{\xi\xi}+4u_{\xi\eta}+u_{\eta\eta}.$$

Substituting into the equation:

$$2(u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta})+3(-2u_{\xi\xi}-3u_{\xi\eta}-u_{\eta\eta})$$

$$+\ 4u_{\xi\xi}+4u_{\xi\eta}+u_{\eta\eta}+7(-u_\xi-u_\eta)+4\,(2u_\xi+u_\eta)-2u=0$$

$$(\underbrace{2-6+4}_{0})u_{\xi\xi}+(4-9+4)u_{\xi\eta}+(\underbrace{2-3+1}_{0})u_{\eta\eta}$$

$$+(-7+8)u_\xi+(-7+4)u_\eta-2u=0$$

$$-u_{\xi\eta}+u_\xi-3u_\eta-2u=0,$$

or $\quad \underline{u_{\xi\eta}+3u_\eta-u_\xi+2u=0}\quad$ is the normal form.

2) $\quad x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}-2yu_x+y\,e^{y/x}=0$

$a=x^2,\quad b=xy,\quad c=y^2$

$\Delta=b^2-ac=x^2y^2-x^2y^2=0$

$\Rightarrow$ PDE is parabolic.

Determining ξ:

$$x^2\phi_x+xy\,\phi_y=0$$

$$\phi_x+\frac{y}{x}\phi_y=0,$$

or $$\frac{dy}{dx}=\frac{y}{x}$$

$$\int\frac{dy}{y}=\int\frac{dx}{x}$$

$$\ln y=\ln x+\underbrace{\ln C}_{\text{arb. const}}$$

$$y=x\cdot C$$

$$\frac{y}{x}=C\quad\Longrightarrow\quad\underline{\xi=\frac{y}{x}}.$$

η can be chosen arbitrarily, independent of ξ. Since the last two terms in the PDE contain y in the coefficients, let us choose $\underline{\eta = y}$.

Derivatives: $$u_x = u_\xi\left(-\frac{y}{x^2}\right) + u_\eta \cdot 0$$

$$u_{xx} = u_{\xi\xi}\left(-\frac{y}{x^2}\right)^2 + u_\xi\left(\frac{2y}{x^3}\right)$$

$$u_{xy} = u_{\xi\xi}\left(-\frac{y}{x^2}\right)\cdot\frac{1}{x} + u_{\xi\eta}\left(-\frac{y}{x^2}\cdot 1 + 0\right) + u_{\eta\eta}\cdot 0$$
$$+ u_\xi\left(-\frac{1}{x^2}\right) + u_\eta \cdot 0$$

$$u_{yy} = u_{\xi\xi}\left(\frac{1}{x}\right)^2 + 2u_{\xi\eta}\frac{1}{x}\cdot 1 + u_{\eta\eta} + 0 .$$

Substituting these into the equation:

$$x^2\left(+\frac{y^2}{x^4}u_{\xi\xi} + \frac{2y}{x^3}u_\xi\right) + 2xy\left(-\frac{y}{x^3}u_{\xi\xi} - \frac{y}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\xi\right)$$
$$+ y^2\left(\frac{1}{x^2}u_{\xi\xi} + \frac{2}{x}u_{\xi\eta} + u_{\eta\eta}\right) + 2y\frac{y}{x^2}u_\xi + y e^{y/x} = 0$$

$$\underbrace{\left(\frac{y^2}{x^2} - \frac{2y^2}{x^2} + \frac{y^2}{x^2}\right)}_{0}u_{\xi\xi} + \underbrace{\left(-\frac{2y^2}{x} + \frac{2y^2}{x}\right)}_{0}u_{\xi\eta} + y^2 u_{\eta\eta}$$
$$+ \frac{2y^2}{x^2}u_\xi + y e^{y/x} = 0$$

$$y^2 u_{\eta\eta} + \frac{2y^2}{x^2}u_\xi + y e^{y/x} = 0$$

Replacing x and y here in terms of ξ and η, we have: $\eta^2 u_{\eta\eta} + 2\xi^2 u_\xi + \eta e^{\xi} = 0$, and in the normal form: $$\underline{u_{\eta\eta} + 2\frac{\xi^2}{\eta^2}u_\xi + \frac{1}{\eta}e^{\xi} = 0} .$$

(2) $u_{xx} + x u_{yy} = 0$

$a = 1, \quad b = 0, \quad c = x$

$\Delta = b^2 - ac = -x$

Hence:

1) $x < 0 \qquad \Delta > 0 \qquad$ hyperbolic

2) $x > 0 \qquad \Delta < 0 \qquad$ elliptic

[3) $x = 0 \qquad \Delta = 0 \qquad$ parabolic: $u_{xx} = 0$]

1/ Let us transform the equation into the normal form for $x < 0$.

Auxiliary equation: $\lambda^2 + x = 0$

$$\lambda^2 = -x \qquad (-x > 0)$$

$$\lambda_{1,2} = \pm\sqrt{-x}$$

Finding the new variables:

$\phi_x - \lambda_1 \phi_y = 0$

is $\phi_x - \sqrt{-x}\,\phi_y = 0$,

so we need to solve

$$\frac{dy}{dx} = -\sqrt{-x}$$

$$\int dy = -\int \sqrt{-x}\,dx$$

$$y = +\frac{2}{3}(-x)^{3/2} + C$$

$$y - \frac{2}{3}(-x)^{3/2} = C$$

$$\xi = y - \frac{2}{3}(-x)^{3/2}.$$

$\psi_x - \lambda_2 \psi_y = 0$

$\psi_x + \sqrt{-x}\,\psi_y = 0$

Solving

$$\frac{dy}{dx} = \sqrt{-x}$$

$$\int dy = \int \sqrt{-x}\,dx$$

$$y = -\frac{2}{3}(-x)^{3/2} + C$$

$$y + \frac{2}{3}(-x)^{3/2} = C$$

$$\eta = y + \frac{2}{3}(-x)^{3/2}$$

$$u_{xx} = u_{\xi\xi}\left(\sqrt{-x}\right)^2 + 2u_{\xi\eta}\left(-\left(\sqrt{-x}\right)^2\right) + u_{\eta\eta}\left(-\sqrt{-x}\right)^2$$
$$+ u_\xi\left(-\frac{1}{2\sqrt{-x}}\right) + u_\eta\left(\frac{1}{2\sqrt{-x}}\right)$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Substituting into the equation:

$$\cancel{u_{\xi\xi}(-x)} - 2u_{\xi\eta}(-x) + \cancel{u_{\eta\eta}(-x)} - \frac{1}{2\sqrt{-x}}u_\xi + \frac{1}{2\sqrt{-x}}u_\eta$$
$$+ x\left(\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}\right) = 0$$

$$4x\,u_{\xi\eta} + \frac{1}{2\sqrt{-x}}\left(u_\eta - u_\xi\right) = 0$$

$$u_{\xi\eta} - \frac{1}{8(-x)\sqrt{-x}}\left(u_\eta - u_\xi\right) = 0$$

$(-x)^{3/2}$ can be expressed by taking the difference $\eta - \xi = \frac{4}{3}(-x)^{3/2}$

$$\Rightarrow (-x)^{3/2} = \frac{3(\eta-\xi)}{4},$$

so the differential equation becomes:

$$u_{\xi\eta} - \frac{1}{8\cdot\frac{3}{4}(\eta-\xi)}\left(u_\eta - u_\xi\right) = 0,$$

or
$$\underline{u_{\xi\eta} + \frac{1}{6(\eta-\xi)}\left(u_\xi - u_\eta\right) = 0.}$$

(normal form for a hyperbolic PDE).

2) $x > 0, \quad \Delta < 0$ (elliptic case).

$$\lambda^2 + x = 0$$

$$\lambda^2 = -x$$

$$\lambda_{1,2} = \pm i\sqrt{x}$$

Let us introduce complex ξ by solving

$$\phi_x - \lambda_1 \phi_y = 0$$

$$\phi_x - i\sqrt{x}\,\phi_y = 0,$$

or $$\frac{dy}{dx} = -i\sqrt{x}$$

$$\int dy = \int -i\sqrt{x}\,dx$$

$$y = -i\frac{2}{3}x^{3/2} + C$$

$$y + i\frac{2}{3}x^{3/2} = C \quad \Rightarrow \quad \text{choose } \underline{\xi = y + i\frac{2}{3}x^{3/2}},$$

and choose $\underline{\eta} = \xi^* = \underline{y - i\frac{2}{3}x^{3/2}}$.

Transforming the derivatives, we obtain:

$$u_{xx} = u_{\xi\xi}(i\sqrt{x})^2 + 2(i\sqrt{x}(-i\sqrt{x}))u_{\xi\eta} + u_{\eta\eta}(-i\sqrt{x})^2$$

$$+ u_\xi\left(\frac{i}{2\sqrt{x}}\right) + u_\eta\left(-\frac{i}{2\sqrt{x}}\right)$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Substituting into the equation:

$$\cancel{u_{\xi\xi}(-x)} + 2x\,u_{\xi\eta} + \cancel{u_{\eta\eta}(-x)} + \frac{i}{2\sqrt{x}}(u_\xi - u_\eta)$$

$$+ x(\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) = 0$$

$$4x\,u_{\xi\eta} + \frac{i}{2\sqrt{x}}(u_\xi - u_\eta) = 0$$

$$u_{\xi\eta} + \frac{i}{8\,x^{3/2}}\,(u_\xi - u_\eta) = 0$$

$$\xi - \eta = i\,\frac{4}{3}\,x^{3/2} \quad \Rightarrow \quad x^{3/2} = \frac{3}{4i}(\xi - \eta)$$

So:

$$u_{\xi\eta} + \frac{i}{8\cdot\frac{3}{4i}(\xi-\eta)}(u_\xi - u_\eta) = 0$$

$$u_{\xi\eta} - \frac{1}{6(\xi-\eta)}(u_\xi - u_\eta) = 0 .$$

This is a complex normal form in the elliptic case, since

$$\xi = y + i\,\frac{2}{3}\,x^{3/2}$$
$$\eta = y - i\,\frac{2}{3}\,x^{3/2} .$$

To obtain the real normal form we introduce

$$\rho = \frac{\xi + \eta}{2} = y ,$$

and

$$\sigma = \frac{\xi - \eta}{2i} = \frac{2}{3}\,x^{3/2} .$$

The derivatives then are:

$$u_{xx} = u_{\rho\rho}\cdot 0 + 2u_{\rho\sigma}\cdot 0 + u_{\sigma\sigma}(\sqrt{x})^2 + u_\rho\cdot 0 + u_\sigma\,\frac{1}{2\sqrt{x}}$$

$$u_{yy} = u_{\rho\rho} + 2u_{\rho\sigma}\cdot 0 + u_{\sigma\sigma}\cdot 0 + u_\rho\cdot 0 + u_\sigma\cdot 0 .$$

Substituting into the PDE:

$$x\,u_{\sigma\sigma} + u_\sigma\,\frac{1}{2\sqrt{x}} + x\,u_{\rho\rho} = 0$$

$$u_{\rho\rho} + u_{\sigma\sigma} + \frac{1}{2\,x^{3/2}}\,u_\sigma = 0 \qquad\qquad x^{3/2} = \frac{3\sigma}{2}$$

$$u_{\rho\rho} + u_{\sigma\sigma} + \frac{1}{3\sigma}\,u_\sigma = 0 .$$

Normal form for an elliptic PDE.

3 $y(x-y^2)(4y^2u_{xx}-u_{yy}) = 8y^3u_x + (5y^2-x)u_y$

Boundary conditions: $u(0,y) = -y^2$, $u_x(0,y) = 1$.

Consider the part with 2nd-order derivatives:

$$4y^2u_{xx} - u_{yy},$$

so $a = 4y^2$, $b = 0$, $c = -1$

$$\Delta = b^2 - ac = 0 + 4y^2 > 0,$$

hence the equation is hyperbolic.

Auxiliary equation:

$$4y^2\lambda^2 - 1 = 0$$

$$\lambda^2 = \frac{1}{4y^2}$$

$$\lambda_{1,2} = \pm\frac{1}{2y}.$$

New variables:

$$\varphi_x - \lambda_1\varphi_y = 0$$

$$\varphi_x - \frac{1}{2y}\varphi_y = 0$$

$$\frac{dy}{dx} = -\frac{1}{2y}$$

$$\int 2y\,dy = \int -dx$$

$$y^2 = -x + C$$

$$y^2 + x = C$$

$$\Rightarrow \xi = x + y^2$$

$$\psi_x - \lambda_2\psi_y = 0$$

$$\psi_x + \frac{1}{2y}\psi_y = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

$$\int 2y\,dy = \int dx$$

$$y^2 = x + C$$

$$x - y^2 = -C$$

Let's use

$$\eta = x - y^2.$$

Transforming the derivatives:

$$u_x = u_\xi + u_\eta$$

$$u_y = u_\xi 2y + u_\eta(-2y) = 2y u_\xi - 2y u_\eta$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi}(2y)^2 + 2u_{\xi\eta}(2y)(-2y) + u_{\eta\eta}(-2y)^2$$

$$+ u_\xi \cdot 2 + u_\eta(-2).$$

Substituting these into the equation, we have:

$$y(x-y^2)\Big[4y^2(\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) - \cancel{u_{\xi\xi}4y^2} + 8y^2 u_{\xi\eta}$$

$$- \cancel{4y^2 u_{\eta\eta}} - 2u_\xi + 2u_\eta\Big] = 8y^3(u_\xi + u_\eta)$$

$$+ (5y^2 - x)(2y u_\xi - 2y u_\eta)$$

$$y(x-y^2)\left[16y^2 u_{\xi\eta} - 2u_\xi + 2u_\eta\right] = 8y^3(u_\xi + u_\eta)$$

$$+ 2y(5y^2 - x)(u_\xi - u_\eta)$$

Dividing both sides by y:

$$(x-y^2)\left[16y^2 u_{\xi\eta} - 2u_\xi + 2u_\eta\right] = 8y^2(u_\xi + u_\eta)$$

$$+ 2(5y^2 - x)(u_\xi - u_\eta)$$

Replacing the x and y variables: $x - y^2 = \eta$

$$2y^2 = \xi - \eta$$

$$2x = \xi + \eta$$

$$\eta\left[8(\xi-\eta)u_{\xi\eta} - 2u_\xi + 2u_\eta\right] = 4(\xi-\eta)(u_\xi + u_\eta)$$

$$+ \left(5(\xi-\eta) - (\xi+\eta)\right)(u_\xi - u_\eta)$$

$$8\eta(\xi-\eta)u_{\xi\eta} - 2\eta u_\xi + \cancel{2\eta u_\eta} = 4\xi u_\xi - 4\eta u_\xi$$
$$+\cancel{4\xi u_\eta} - \cancel{4\eta u_\eta} + 4\xi u_\xi - 6\eta u_\xi - \cancel{4\xi u_\eta} + \cancel{6\eta u_\eta}$$

$$8\eta(\xi-\eta)u_{\xi\eta} + 8\eta u_\xi - 8\xi u_\xi = 0$$

$$8\eta(\xi-\eta)u_{\xi\eta} = 8(\xi-\eta)u_\xi$$

For $\xi \neq \eta$ $\qquad \eta u_{\xi\eta} = u_\xi$

$$\underline{u_{\xi\eta} - \frac{1}{\eta}u_\xi = 0} : \quad \underline{\text{normal form.}}$$

Let $u_\xi = v$. The above equation then reads:

$$v_\eta - \frac{v}{\eta} = 0$$

$$\frac{\partial v}{\partial \eta} = \frac{v}{\eta} \quad : \text{ for } \xi = \text{const}$$
$\partial \to d$ and integrate

$$\int \frac{dv}{v} = \int \frac{d\eta}{\eta}$$

constant may depend on ξ.

$$\ln v = \ln \eta + C(\xi)$$

Taking exp of both sides: $v = \underbrace{e^{C(\xi)}}_{f(\xi)} \eta$

$f(\xi)$ - arbitrary function

$$\Rightarrow \quad v = \eta f(\xi)$$

$$u_\xi = \eta f(\xi)$$

Integrating W.R.T. ξ we have:

$$u = \eta \underbrace{\int f(\xi)\, d\xi}_{\text{function of } \xi, \text{ denote } g(\xi)} + \underbrace{h(\eta)}_{}$$

arb. "constant" (can depend on η)

Therefore, the general solution of the PDE is

$$u = \eta\, g(\xi) + h(\eta),$$

or, in terms of x and y:

$$\underline{u = (x-y^2)\, g(x+y^2) + h(x-y^2)}. \quad (*)$$

Applying the boundary condition, $u(0,y) = -y^2$, by setting $x=0$ in $(*)$:

$$-y^2 g(y^2) + h(-y^2) = -y^2.$$

To apply the end boundary condition,

$$u_x = g(x+y^2) + (x-y^2)\, g'(x+y^2) + h'(x-y^2),$$

and for $x=0$:

$$g(y^2) - y^2 g'(y^2) + h'(-y^2) = 1.$$

So, we have two conditions to find g and h:

$$\begin{cases} -y^2 g(y^2) + h(-y^2) = -y^2 \\ g(y^2) - y^2 g'(y^2) + h'(-y^2) = 1 \end{cases}$$

Denote $y^2 = z$. We then have:

$$\begin{cases} -z\, g(z) + h(-z) = -z & (1) \\ g(z) - z\, g'(z) + h'(-z) = 1. & (2) \end{cases}$$

Differentiating the 1st of these equations WRT z,

$$-g(z) - z\, g'(z) - h'(-z) = -1$$

Adding this to (2), we have:

$$-2z\, g'(z) = 0 \Rightarrow g'(z) = 0 \Rightarrow \underline{g = A}$$

Substituting into (1): $-zA + h(-z) = -z$

$$h(-z) = (-z)(1-A)$$

$$\Rightarrow \underline{h(z) = z(1-A)}.$$

Substituting g and h into $(*)$:

$$u = (x-y^2)\cdot A + (x-y^2)(1-A)$$

$$\underline{u = x-y^2} .$$