AMA3006 Problem sheet 9

Homework problems

SOLUTIONS

(1) (a) $u_{xx} - 2u_{xy} + u_{yy} + 9u_x + 9u_y - 9u = 0$

$a = 1, \quad b = -1, \quad c = 1$

$\Delta = b^2 - ac = 1 - 1 = 0 \Rightarrow$ equation is <u>parabolic</u>.

To find new variable $\xi = \phi(x,y)$, we solve

$a\phi_x + b\phi_y = 0 \iff \phi_x - \phi_y = 0.$

To solve this, we need to solve the ODE

$$\frac{dy}{dx} = -1$$

$$\int dy = -\int dx$$

$$y = -x + C$$

$$y + x = C.$$

Hence $\xi = x + y$.

Let's choose $\eta = x$.

$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + u_\eta$$

$$u_y = u_\xi \cdot 1 + u_\eta \cdot 0 = u_\xi$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = u_{\xi\xi} + u_{\xi\eta}$$

$$u_{yy} = u_{\xi\xi}$$

Substituting into the PDE:

$$\cancel{u_{\xi\xi}} + 2\cancel{u_{\xi\eta}} + u_{\eta\eta} - 2(\cancel{u_{\xi\xi}} + \cancel{u_{\xi\eta}}) + \cancel{u_{\xi\xi}} + 9(u_\xi + u_\eta) + 9u_\xi - 9u = 0.$$

$$\underline{u_{\eta\eta} + 18u_\xi + 9u_\eta - 9u = 0.}$$

(b) $u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 15u_y + 27x = 0$

$a = 1, \quad b = \frac{1}{2}, \quad c = -2$

$\Delta = \left(\frac{1}{2}\right)^2 + 2 = \frac{9}{4} > 0$: equation is <u>hyperbolic</u>.

Auxiliary equation: $\lambda^2 + \lambda - 2 = 0$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$

$$\lambda_1 = 1, \quad \lambda_2 = -2.$$

Finding new variables:

ξ	η
$\phi_x - \phi_y = 0$	$\psi_x + 2\psi_y = 0$
$\frac{dy}{dx} = -1$	$\frac{dy}{dx} = 2$
$\int dy = \int -dx$	$\int dy = \int 2\,dx$
$y = -x + C$	$y = 2x + C$
$y + x = C$	$2x - y = -C$
$\underline{\xi = x + y}$.	$\underline{\eta = 2x - y}$.
	(another possible choice could be $\eta = y - 2x$)

$u_x = u_\xi \cdot 1 + u_\eta \cdot 2 = u_\xi + 2u_\eta$

$u_y = u_\xi \cdot 1 + u_\eta(-1) = u_\xi - u_\eta$

$u_{xx} = u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta}$

$u_{xy} = u_{\xi\xi} + u_{\xi\eta}(-1+2) - 2u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

Substituting these into the PDE,

$$\cancel{u_{\xi\xi}} + 4u_{\xi\eta} + 4\cancel{u_{\eta\eta}} + \cancel{u_{\xi\xi}} + u_{\xi\eta} - 2\cancel{u_{\eta\eta}} - 2(\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}})$$
$$-3(u_\xi + 2u_\eta) - 15(u_\xi - u_\eta) + 27x = 0$$

$$9u_{\xi\eta} - 18u_\xi + 9u_\eta + 27\,\frac{\xi+\eta}{3} = 0$$

$\xi + \eta = 3x \Rightarrow x = \dfrac{\xi+\eta}{3}$

$$9u_{\xi\eta} - 18u_\xi + 9u_\eta + 9(\xi+\eta) = 0$$

$$\underline{u_{\xi\eta} - 2u_\xi + u_\eta + \xi + \eta = 0.}$$

(c) $u_{xx} + 2u_{xy} + 5u_{yy} - 32u = 0$

$a = 1,\quad b = 1,\quad c = 5$

$\Delta = 1 - 5 = -4$: equation is elliptic.

Auxiliary equation: $\lambda^2 + 2\lambda + 5 = 0$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-5}}{1} = \frac{-1 \pm 2i}{1}$$

$$\lambda_1 = -1 + 2i, \qquad \lambda_2 = -1 - 2i$$

Finding new variables:

ξ

$\phi_x - (-1+2i)\phi_y = 0$

$\dfrac{dy}{dx} = 1 - 2i$

$\int dy = \int (1-2i)\,dx$

$y - (1-2i)x = C$

$\xi = y - (1-2i)x$

η

$\psi_x - (-1-2i)\psi_y = 0$

$\dfrac{dy}{dx} = 1 + 2i$

$\int dy = \int (1+2i)\,dx$

$y - (1+2i)x = C$

$\eta = y - (1+2i)x$

Note: $\eta = \xi^*$

In these complex variables, we have:

$$u_{xx} = u_{\xi\xi}(1-2i)^2 + 2u_{\xi\eta}(1-2i)(1+2i) + u_{\eta\eta}(1+2i)^2$$

$$u_{xy} = u_{\xi\xi}(-1+2i) + u_{\xi\eta}(-1-2i-1+2i) + u_{\eta\eta}(-1-2i)$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Substituting into the PDE and regrouping, we have:

$$u_{\xi\xi}\left[(1-2i)^2+2(-1+2i)+5\right]$$
$$+u_{\xi\eta}\left[2(1-2i)(1+2i)+2(-2)+10\right]$$
$$+u_{\eta\eta}\left[(1+2i)^2+2(-1-2i)+5\right]-32u=0$$

$$u_{\xi\xi}\left[1-4i-4-2+4i+5\right]+u_{\xi\eta}\left[2+8-4+10\right]$$
$$+u_{\eta\eta}\left[1+4i-4-2-4i+5\right]-32u=0$$
$$16u_{\xi\eta}-32u=0$$
$$\underline{u_{\xi\eta}-2u=0.}$$ (Normal form in complex ξ and η)

For the real normal form, introduce

$$\rho=\frac{\xi+\eta}{2}=\frac{y-(1-2i)x+y-(1+2i)x}{2}=y-x$$
$$\sigma=\frac{\xi-\eta}{2i}=\frac{2ix+2ix}{2i}=2x.$$

$$u_{xx}=u_{\rho\rho}(-1)^2+2u_{\rho\sigma}\cdot(-2)+u_{\sigma\sigma}\cdot 2^2$$
$$u_{xy}=u_{\rho\rho}(-1)+u_{\rho\sigma}(2+0)+u_{\sigma\sigma}\cdot 0$$
$$u_{yy}=u_{\rho\rho}\cdot 1$$

Substituting into the PDE:

$$u_{\rho\rho}-\cancel{4u_{\rho\sigma}}+4u_{\sigma\sigma}-2u_{\rho\rho}+\cancel{4u_{\rho\sigma}}+5u_{\rho\rho}-32u=0$$
$$4u_{\rho\rho}+4u_{\sigma\sigma}-32u=0$$
$$\underline{u_{\rho\rho}+u_{\sigma\sigma}-8u=0.}$$ (The real normal form.)

(2) $u_{xx} + y\, u_{yy} = 0$

$a = 1, \quad b = 0, \quad c = y$

$\Delta = -y$, so $\Delta > 0$ for $y < 0$ (hyperbolic)

and $\Delta < 0$ for $y > 0$ (elliptic).

[For $y = 0$, $\Delta = 0$, equation is $u_{xx} = 0$ – parabolic.]

1) $y < 0$: auxiliary equation

$$\lambda^2 + y = 0$$

$$\lambda^2 = -y$$

$$\lambda_{1,2} = \pm\sqrt{-y}$$

New variables:

ξ	η
$\phi_x + \sqrt{-y}\, \phi_y = 0$	$\psi_x - \sqrt{-y}\, \psi_y = 0$
$\frac{dy}{dx} = +\sqrt{-y}$	$\frac{dy}{dx} = -\sqrt{-y}$
$\int \frac{dy}{\sqrt{-y}} = \int dx$	$\int \frac{dy}{\sqrt{-y}} = \int - dx$
$-2\sqrt{-y} = x + C$	$-2\sqrt{-y} = -x + C$
$x + 2\sqrt{-y} = -C$	$x - 2\sqrt{-y} = C$
$\underline{\xi = x + 2\sqrt{-y}}$	$\underline{\eta = x - 2\sqrt{-y}}$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi}\left(-\frac{1}{\sqrt{-y}}\right)^2 + 2u_{\xi\eta}\frac{1}{\sqrt{-y}}\left(-\frac{1}{\sqrt{-y}}\right) + u_{\eta\eta}\left(\frac{1}{\sqrt{-y}}\right)^2 + u_\xi \frac{-1}{2(-y)^{3/2}}$$
$$+ u_\eta\left(-\frac{-1}{2(-y)^{3/2}}\right)$$

Substituting into the PDE:

$$\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}} + y\left[\cancel{u_{\xi\xi}\frac{1}{-y}} + \frac{2u_{\xi\eta}}{y} + \cancel{u_{\eta\eta}\frac{1}{-y}} + u_\xi \frac{-1}{2(-y)^{3/2}} + u_\eta \frac{1}{2(-y)^{3/2}}\right] = 0$$

$$4u_{\xi\eta} + \frac{1}{2\sqrt{-y}}u_\xi - \frac{1}{2\sqrt{-y}}u_\eta = 0$$

$$\xi - \eta = 4\sqrt{-y} \Rightarrow \frac{1}{2\sqrt{-y}} = \frac{2}{\xi-\eta}.$$

So,: we have: $\underline{4u_{\xi\eta} + \frac{2}{\xi-\eta}(u_\xi - u_\eta) = 0}$.

2) $y > 0$, auxiliary equation:

$$\lambda^2 + y = 0$$
$$\lambda^2 = -y$$
$$\lambda = \pm i\sqrt{y}$$

To find ξ:

$$\phi_x + i\sqrt{y}\,\phi_y = 0$$

$$\frac{dy}{dx} = i\sqrt{y}$$

$$\int \frac{dy}{\sqrt{y}} = \int i\,dx$$

$$2\sqrt{y} = ix + C$$

$$ix - 2\sqrt{y} = -C$$

$$x + 2i\sqrt{y} = -\frac{C}{i}$$

$$\underline{\xi = x + 2i\sqrt{y}}$$

We could have chosen $\xi = 2\sqrt{y} - ix$, but $\xi = x + 2i\sqrt{y}$ looks nicer!

Then $\eta = \xi^* = x - 2i\sqrt{y}$, and the new <u>real</u> variables are $\rho = \frac{\xi+\eta}{2} = x$, $\sigma = \frac{\xi-\eta}{2i} = 2\sqrt{y}$.

$u_{xx} = u_{\rho\rho}$, $u_{yy} = u_{\sigma\sigma}\left(\frac{1}{\sqrt{y}}\right)^2 + u_\sigma\left(-\frac{1}{2y^{3/2}}\right)$.

Substituting into the PDE:

$$u_{\rho\rho} + y\left(u_{\sigma\sigma} \cdot \frac{1}{y} - \frac{1}{2y^{3/2}} u_\sigma \right) = 0$$

$$u_{\rho\rho} + u_{\sigma\sigma} - \frac{1}{2\sqrt{y}} u_\sigma = 0$$

$$\underline{u_{\rho\rho} + u_{\sigma\sigma} - \frac{1}{\sigma} u_\sigma = 0.}$$

(3) $u_{xx} + 2x u_{xy} + x^2 u_{yy} = 0$: $a = 1,\ b = x,\ c = x^2$

$\Delta = x^2 - x^2 = 0$ - the equation is parabolic.

Equation for $\xi = \phi(x,y)$: $a\phi_x + b\phi_y = 0$

$$\phi_x + x\phi_y = 0$$

$$\frac{dy}{dx} = x$$

$$\int dy = \int x\,dx$$

$$y = \frac{x^2}{2} + C$$

$$y - \frac{x^2}{2} = C \quad \Rightarrow \quad \underline{\xi = y - \frac{x^2}{2}.}$$

Choosing $\eta = y + \frac{x^2}{2}$, as suggested, we have:

$$u_{xx} = u_{\xi\xi}(-x)^2 + 2u_{\xi\eta}\, x(-x) + u_{\eta\eta}\, x^2 + u_\xi(-1) + u_\eta \cdot 1$$

$$u_{xy} = u_{\xi\xi}(-x) + u_{\xi\eta}(-x + x) + u_{\eta\eta} \cdot (x)$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Substituting into the equation and rearranging:

$$u_{\xi\xi}(x^2 - 2x^2 + x^2) + u_{\xi\eta}(-2x^2 + 2x^2) + u_{\eta\eta}(x^2 + 2x^2 + x^2) - u_\xi + u_\eta = 0$$

$4x^2 u_{\eta\eta} + u_\eta - u_\xi = 0$; and since $\eta - \xi = x^2$, $\underline{4(\eta - \xi)u_{\eta\eta} + u_\eta - u_\xi = 0.}$

4 $u_{xx} - u_{yy} = 16xy$

$a = 1, \ b = 0, \ c = -1$

$\Delta = 0 + 1 = 1 > 0$: equation is hyperbolic.

Auxiliary equation $\lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm 1$

New variables:

$\phi_x - \phi_y = 0$	$\psi_x + \psi_y = 0$
$\frac{dy}{dx} = -1$	$\frac{dy}{dx} = 1$
$\int dy = \int -dx$	$\int dy = \int dx$
$y = -x + C$	$y = x + C$
$y + x = C$	$y - x = C$
$\Rightarrow \xi = y + x$	$\Rightarrow \eta = y - x$

$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta}(-1) + u_{\eta\eta}(-1)^2$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

To express the right-hand side of the PDE in terms of ξ and η, take squares of ξ and η:

$\xi^2 = y^2 + 2xy + x^2$

$\eta^2 = y^2 - 2xy + x^2$

and subtract these: $\xi^2 - \eta^2 = 4xy$.

Substituting everything onto the PDE,

$\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} - \cancel{u_{\xi\xi}} - 2u_{\xi\eta} - \cancel{u_{\eta\eta}} = 4(\xi^2 - \eta^2)$

$-4u_{\xi\eta} = 4(\xi^2 - \eta^2) \iff u_{\xi\eta} = \eta^2 - \xi^2$.

Integrating this equation with respect to ξ, we obtain:

$$u_\eta = \eta^2\xi - \frac{1}{3}\xi^3 + \underbrace{f(\eta)}_{\substack{\text{arbitrary function} \\ \text{(constant W.R.T. } \xi)}}$$

Integrating this W.R.T η now, we have:

$$u = \frac{1}{3}\eta^3\xi - \frac{1}{3}\eta\xi^3 + \underbrace{\int f(\eta)\,d\eta}_{h(\eta)} + \underbrace{g(\xi)}_{\substack{\text{arbitrary} \\ \text{function}}}$$

Hence: $\underline{u = \frac{1}{3}\xi\eta(\eta^2 - \xi^2) + g(\xi) + h(\eta)}$.

Substituting ξ and η,

$$u(x,y) = \frac{1}{3}(y+x)(y-x)\left[(y-x)^2 - (y+x)^2\right] + g(x+y) + h(y-x)$$

$$u(x,y) = \frac{1}{3}(y^2 - x^2)(-4xy) + g(x+y) + h(y-x)$$

$$u = \frac{4}{3}(x^3y - xy^3) + g(x+y) + h(y-x) \quad (*)$$

Using the boundary conditions,

$u(0,y) = 2y^2$ gives: $\quad g(y) + h(y) = 2y^2 \quad (1)$

$$u_x = 4x^2y - \frac{4}{3}y^3 + g'(x+y) - h'(y-x)$$

and $u_x(0,y) = 0$ gives

$$g'(y) - h'(y) = \frac{4}{3}y^3 \quad (2)$$

Differentiating (1)2 W.R.T. y we have:

$$g'(y) + h'(y) = 4y$$

Adding this to (2) gives:

$$2g'(y) = 4y + \frac{4}{3}y^3$$

$$g'(y) = 2y + \frac{2}{3}y^3$$

Integrating,

$$g(y) = y^2 + \frac{2}{3\cdot 4}y^4 + C$$

$$g(y) = y^2 + \frac{1}{6}y^4 + C.$$

Then, from (1)

$$h(y) = 2y^2 - g(y)$$

$$h(y) = y^2 - \frac{1}{6}y^4 - C.$$

Substituting these into (*) on page 9, we have:

$$u = \frac{4}{3}(x^3y - xy^3) + (x+y)^2 + \frac{1}{6}(x+y)^4 + C$$
$$+ (y-x)^2 - \frac{1}{6}(y-x)^4 - C$$

$$u = \frac{4}{3}(x^3y - xy^3) + x^2 + \cancel{2xy} + y^2 + y^2 - \cancel{2xy} + x^2$$
$$+ \frac{1}{6}\left[\cancel{x^4} + 4x^3y + \cancel{6x^2y^2} + 4xy^3 \cancel{+y^4}\right.$$
$$\left. \cancel{-y^4} + 4y^3x - \cancel{6x^2y^2} + 4yx^3 \cancel{-x^4}\right]$$

$$u = \frac{4}{3}(x^3y - xy^2) + 2x^2 + 2y^2 + \frac{4}{3}xy^3 + \frac{4}{3}yx^3$$

$$\Longrightarrow \quad \underline{u = \frac{8}{3}x^3y + 2x^2 + 2y^2.}$$

5. $u_{xx} + 4u_{yy} = 8xy$

$a = 1,\ b = 0,\ c = 4$

$\Delta = -4 < 0$ - equation is elliptic.

Auxiliary equation: $\lambda^2 + 4 = 0$

$\lambda^2 = -4$

$\lambda_{1,2} = \pm 2i$

Finding complex variables ξ and η:

$$\phi_x - 2i\,\phi_y = 0$$

$$\frac{dy}{dx} = -2i$$

$$\int dy = \int -2i\,dx$$

$$y = -2ix + C$$

$$y + 2ix = C \quad \Rightarrow \quad \xi = y + 2ix$$

$$\eta = \xi^* = y - 2ix$$

$$u_{xx} = u_{\xi\xi}(2i)^2 + 2u_{\xi\eta}\,(2i)(-2i) + u_{\eta\eta}(-2i)^2$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Substituting into the PDE, we have:

$$\cancel{-4u_{\xi\xi}} + 8u_{\xi\eta} - \cancel{4u_{\eta\eta}} + \cancel{4u_{\xi\xi}} + 8u_{\xi\eta} + \cancel{4u_{\eta\eta}} = 8xy$$

$$16u_{\xi\eta} = 8xy$$

To change the right-hand side to ξ and η,

$$\xi^2 = y^2 + 4ixy - 4x^2$$

$$\eta^2 = y^2 - 4ixy - 4x^2 \quad \Rightarrow \quad \xi^2 - \eta^2 = 8ixy$$

$$\Rightarrow \quad 8xy = \frac{1}{i}(\xi^2 - \eta^2) = i(\eta^2 - \xi^2)$$

$$16 u_{\xi\eta} = i(\eta^2 - \xi^2)$$ - complex normal form.

Integrating W.R.T. ξ :

$$16 u_\eta = i\eta^2 \xi - \frac{i}{3}\xi^3 + f(\eta)$$ ← arbitrary function

Integrating W.R.T. η :

$$16 u = \frac{i}{3}\eta^3 \xi - \frac{i}{3}\eta \xi^3 + \underbrace{\int f(\eta)\, d\eta}_{h(\eta)} + g(\xi)$$

$$\Rightarrow \quad u = \frac{i}{48}\xi\eta(\eta^2 - \xi^2) + h(\eta) + g(\xi)$$

(where we have redefined the arbitrary functions $g(\xi)$ and $h(\eta)$).

Changing back to x and y,

$$\eta^2 - \xi^2 = \frac{8xy}{i}$$

$$\xi\eta = (y + 2ix)(y - 2ix) = y^2 + 4x^2$$

$$\Rightarrow \quad u = \frac{i}{48}(y^2 + 4x^2)\frac{8xy}{i} + g(y + 2ix) + h(y - 2ix)$$

$$u = \frac{1}{6}xy(y^2 + 4x^2) + g(y + 2ix) + h(y - 2ix)$$

is the general solution.

Using the boundary conditions:

$u(0,y) = y^2$ gives: $\quad g(y) + h(y) = y^2 \quad (1)$

$$u_x = \frac{1}{6}y^3 + 2yx^2 + 2i\, g'(y + 2ix) - 2i\, h'(y - 2ix)$$

$u_x(0,y) = 0$ gives $\quad g'(y) - h'(y) = \frac{i}{12}y^3 \quad (2)$

Differentiating (1), $g'(y) + h'(y) = 2y$,

and adding this to (2):

$$2g'(y) = 2y + \frac{i}{12}y^3$$

$$g'(y) = y + \frac{i}{24}y^3$$

Integrating:

$$g(y) = \frac{y^2}{2} + \frac{i}{96}y^4 + C$$

And then, using (1):

$$h(y) = y^2 - g(y)$$

$$h(y) = \frac{y^2}{2} - \frac{i}{96}y^4 - C$$

Substituting g and h into the general solution,

$$u = \frac{1}{6}xy(y^2+4x^2) + \frac{1}{2}(y+2ix)^2 + \frac{i}{96}(y+2ix)^4 + C$$
$$+ \frac{1}{2}(y-2ix)^2 - \frac{i}{96}(y-2ix)^4 - C$$

$$u = \frac{1}{6}xy(y^2+4x^2) + \frac{1}{2}(y^2+y^2-4x^2-4x^2)$$

$$+ \frac{i}{96}\left(\cancel{y^4} + 8iy^3x - \cancel{24y^2x^2} - 32yix^3 + \cancel{16x^4}\right.$$
$$\left. - \cancel{y^4} + 8iy^3x + \cancel{24y^2x^2} - 32yix^3 - \cancel{16x^4}\right)$$

Here we use the binomial expansion

$$u = \frac{1}{6}xy(y^2+4x^2) + y^2 - 4x^2 - \frac{1}{6}y^3x + \frac{2}{3}yx^3$$

$$\underline{u = \frac{4}{3}x^3y + y^2 - 4x^2.}$$

⑥ $$y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$$

$a = y^2, \quad b = -xy, \quad c = x^2$

$\Delta = x^2y^2 - x^2y^2 = 0 \Rightarrow$ equation is <u>parabolic</u>.

Choosing ξ:
$$a\phi_x + b\phi_y = 0$$
$$y^2\phi_x - xy\,\phi_y = 0$$
$$\phi_x - \frac{x}{y}\phi_y = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$
$$y\,dy = -x\,dx$$
$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$
$$y^2 + x^2 = 2C$$
$$\Rightarrow \quad \xi = x^2 + y^2$$

Let us choose $\eta = x$.

$$u_{xx} = u_{\xi\xi}(2x)^2 + 2u_{\xi\eta}(2x) + u_{\eta\eta} + u_\xi 2$$
$$u_{xy} = u_{\xi\xi}(2x\,2y) + u_{\xi\eta}(2x\cdot 0 + 2y\cdot 1) + u_{\eta\eta}\cdot 0$$
$$u_{yy} = u_{\xi\xi}(2y)^2 + 2u_{\xi\eta}\cdot 0 + u_{\eta\eta}\cdot 0 + u_\xi\cdot 2$$

First derivatives: $u_x = u_\xi \cdot 2x + u_\eta$

$u_y = u_\xi \cdot 2y + u_\eta \cdot 0$

Substituting into the PDE, we have,

$$u_{\xi\xi}(4x^2y^2 - 8x^2y^2 + 4x^2y^2) + u_{\xi\eta}(4xy^2 - 4xy^2) + 2x^2u_\xi$$
$$+ u_{\eta\eta}(y^2) + 2y^2u_\xi = \frac{y^2}{x}(2xu_\xi + u_\eta) + \frac{x^2}{y}2yu_\xi$$

$$y^2 u_{\eta\eta} + \cancel{2x^2 u_\xi} + \cancel{2y^2 u_\xi} = \cancel{2y^2 u_\xi} + \frac{y^2}{x} u_\eta + \cancel{2x^2 u_\xi}$$

$$y^2 u_{\eta\eta} - \frac{y^2}{x} u_\eta = 0$$

$$u_{\eta\eta} - \frac{1}{\eta} u_\eta = 0 .$$

Multiply by $\frac{1}{\eta}$:

$\frac{1}{\eta}$ here plays the role of an integrating factor

$$\frac{1}{\eta} u_{\eta\eta} - \frac{1}{\eta^2} u_\eta = 0$$

$$\frac{\partial}{\partial \eta} \left(\frac{u_\eta}{\eta} \right) = 0$$

Alternatively: introduce $v = u_\eta$, so that the PDE becomes:

$$v_\eta - \frac{1}{\eta} v = 0$$

and solve this as a 1st-order linear ODE for v, remembering that an arbitrary const can depend on ξ.

Integrating: $\frac{u_\eta}{\eta} = f(\xi)$ ← arbitrary function

$$u_\eta = \eta f(\xi)$$

Integrating again:

$$u = \frac{\eta^2}{2} f(\xi) + g(\xi)$$

↖ another arbitrary function.

Redefining $\frac{1}{2} f(\xi) \to f(\xi)$, we can write the general solution as

$$u = \eta^2 f(\xi) + g(\xi) .$$

Substituting ξ and η, we obtain:

$$\underline{u = x^2 f(x^2+y^2) + g(x^2+y^2)} .$$

(7) $u_{xx} + 2\sin x\, u_{xy} - (\cos^2 x - \sin^2 x) u_{yy} + \cos x\, u_y = 0$

$a = 1, \quad b = \sin x, \quad c = -(\cos^2 x - \sin^2 x)$

$\Delta = \sin^2 x + \cos^2 x - \sin^2 x = \cos^2 x$

$\Delta > 0$ for $\cos x \neq 0$, i.e. $x \neq \frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$

The equation is hyperbolic for $x = \pi(n + \frac{1}{2})$.

Auxiliary equation:

$$\lambda^2 + 2\sin x\, \lambda - (\cos^2 x - \sin^2 x) = 0$$

$$\lambda_{1,2} = \frac{-\sin x \pm \sqrt{\sin^2 x + \cos^2 x - \sin^2 x}}{1}$$

$$\lambda_{1,2} = -\sin x \pm \cos x.$$

New variables:

ξ	η
$\phi_x - (-\sin x + \cos x)\phi_y = 0$	$\psi_x - (-\sin x - \cos x)\psi_y = 0$
$\frac{dy}{dx} = \sin x - \cos x$	$\frac{dy}{dx} = \sin x + \cos x$
$\int dy = \int (\sin x - \cos x)\, dx$	$\int dy = \int (\sin x + \cos x)\, dx$
$y = -\cos x - \sin x + C$	$y = -\cos x + \sin x + C$
$y + \cos x + \sin x = C$	$y + \cos x - \sin x = C$
$\Rightarrow \xi = y + \cos x + \sin x.$	$\Rightarrow \eta = y + \cos x - \sin x.$

$$u_{xx} = u_{\xi\xi}(-\sin x+\cos x)^2 + 2u_{\xi\eta}(-\sin x+\cos x)(-\sin x-\cos x)$$
$$+u_{\eta\eta}(-\sin x-\cos x)^2 + u_\xi(-\cos x-\sin x)$$
$$+u_\eta(-\cos x+\sin x)$$

$$u_{xy} = u_{\xi\xi}(-\sin x+\cos x) + u_{\xi\eta}(-\sin x+\cancel{\cos x}-\sin x-\cancel{\cos x})$$
$$+u_{\eta\eta}(-\sin x-\cos x)$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

And the first derivative: $u_y = u_\xi + u_\eta$

Substituting these into the PDE and rearranging:

$$u_{\xi\xi}\big(\sin^2 x - 2\sin x\cos x + \cos^2 x + 2\sin x(-\sin x+\cos x)$$
$$-(\cos^2 x-\sin^2 x)\big)$$
$$+u_{\xi\eta}\big(2\sin^2 x - 2\cos^2 x - 4\sin^2 x - 2(\cos^2 x - \sin^2 x)\big)$$
$$+u_{\eta\eta}\big(\sin^2 x + 2\sin x\cos x + \cos^2 x - 2\sin x(\sin x+\cos x)$$
$$-(\cos^2 x-\sin^2 x)\big)$$
$$+u_\xi(-\cos x-\sin x) + u_\eta(-\cos x+\sin x) + \cos x(u_\xi+u_\eta) = 0$$

$$-4\cos^2 x\, u_{\xi\eta} - \sin x\, u_\xi + \sin x\, u_\eta = 0$$

$$4\cos^2 x\, u_{\xi\eta} + \sin x\,(u_\xi - u_\eta) = 0 \qquad (*)$$

$$\xi - \eta = 2\sin x \quad \Rightarrow \quad \sin x = \tfrac{1}{2}(\xi-\eta)$$

$$\cos^2 x = 1-\sin^2 x = 1 - \tfrac{1}{4}(\xi-\eta)^2$$

Substituting these into $(*)$ we have:

$$4\left(1-\frac{1}{4}(\xi-\eta)^2\right)u_{\xi\eta}+\frac{1}{2}(\xi-\eta)(u_\xi-u_\eta)=0$$

$$u_{\xi\eta}+\frac{\xi-\eta}{2\left[4-(\xi-\eta)^2\right]}(u_\xi-u_\eta)=0.$$