Unusual statistics of interference effects in neutron scattering from compound nuclei

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We consider interference effects between *p*-wave resonance scattering amplitude and background *s*-wave amplitude in low-energy neutron scattering from a heavy nucleus which goes through the compound nucleus stage. The first effect is in the difference between the forward and backward scattering cross sections (the $\mathbf{p}_i \cdot \mathbf{p}_f$ correlation). Because of the chaotic nature of the compound states, this effect is a random variable with zero mean. However, a statistical consideration shows that the probability distribution of this effect does not obey the standard central limit theorem. That is, the probability density for the effect averaged over *n* resonances does not become a Gaussian distribution with the variance decreasing as $n^{-1/2}$ (''violation'' of the theorem). We derive the probability distribution of the effect and the limit distribution of the average. It is found that the width of this distribution does not decrease with the increase of *n*, i.e., fluctuations are not suppressed by averaging. Furthermore, we consider the $\boldsymbol{\sigma}\cdot(\mathbf{p}_i \times \mathbf{p}_f)$ correlation and find that this effect, although much smaller, shows fluctuations which actually increase upon averaging over many measurements. This behavior holds for $\epsilon > \Gamma_p$ where ϵ is the distance to the resonance, and Γ_p is the resonance width. Limits of the effects due to finite resonance widths are also considered. In the Appendix we present a simple derivation of the limit theorem for the average of random variables with infinite variances.

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I. INTRODUCTION

Due to chaotic nature of compound nuclei, positions of *s* and *p* resonances in neutron scattering from a heavy nucleus, and amplitudes involving these states, are uncorrelated. This gives rise to an unusual statistical effect in the asymmetry of the transmission of neutrons with positive and negative helicities [1]. This asymmetry corresponds to the $\boldsymbol{\sigma}\cdot\mathbf{p}$ correlation. It violates parity conservation, and is produced by the weak interaction in the nucleus, which mixes the *s* and *p* neutron partial waves. The magnitude of the asymmetry is strongly enhanced if the neutron energy is tuned into the *p* resonance [2]. In this case its magnitude is determined by the perturbative mixing

$$\eta = \frac{\langle s|W|p\rangle}{E_s - E_p} \tag{1}$$

of the *s* and *p* resonances by the weak interaction *W*. The matrix element between the compound states behaves as a Gaussian random variable, and η is also a random variable with zero mean. The characteristic mixing (and asymmetry) can be estimated simply as $\eta_c \sim w/D$, where *w* is the root-mean-square matrix element, and *D* is the mean spacing between the compound nucleus resonances. However, when one takes a closer look at the probability distribution of η, it turns out that its variance is infinite. This effect originates from a high probability to find small spacings $E_s - E_p$, which results in the slow decrease of the probability density, $f(\eta) \propto 1/\eta^2$.

The standard central limit theorem (CLT) is not applicable to such random variables. Instead, it can be shown that if we consider the statistics of the average of *n* such variables, the probability density of the sum $(\eta_1 + \cdots + \eta_n)/n$ becomes independent of *n* at large *n*, i.e., fluctuations of η are not averaged out [3]. This contrasts the usual situation where CLT would give a Gaussian distribution whose width decreases as $1/\sqrt{n}$. This unusual behavior is explained by the fact that among *n* uncorrelated η_i there is a large probability to find one, whose magnitude is *n* times greater than their typical value, $\eta_i \sim n\eta_c$. Such η_i will always dominate the sum and ensure that fluctuations are not suppressed by averaging. Another physical instance in which rare events dominate the distribution is seen in ''Levy flights'' in the random force diffusion model [4]. In a usual homogeneous system, diffusion is modeled by Brownian motion, where the distribution of displacements is Gaussian. But in the random force diffusion model, disorder induces rare but large displacements which dominate the distribution (the Levy distribution). Under certain values of parameters, these have statistics similar to those found in nuclear scattering problems.

General statistical properties of nuclear cross sections were considered by Ericson [5]. In this paper we analyze effects in neutron scattering that have unusual statistics. They are more conventional than those discussed in Ref. [1], since they do not involve the weak interaction. For example, in Sec. II we consider the difference between the differential cross-section in the forward and backward scattering, due to the interference of the *p*-wave resonance scattering and the background scattering amplitude, for a spinless particle. This effect can be described as the $\mathbf{p}_i \cdot \mathbf{p}_f$ correlation, where $\mathbf{p}_i$ and $\mathbf{p}_f$ are the momentum of the incident and emitted particles, respectively. We derive the statistics of the observable and the way it behaves upon averaging over many different nuclei, in a similar fashion to what was done in [1]. We also discuss the limit of this effect when finite widths of the reso-

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nances are taken into account. In Sec. III we rederive the $\mathbf{p}_i \cdot \mathbf{p}_f$ correlation for particles with spin (e.g., neutrons) incident upon a spinless nucleus, as well as study the effect of a different correlation between the neutron spin $\boldsymbol{\sigma}$ and the scattering plane, $\boldsymbol{\sigma} \cdot (\mathbf{p}_i \times \mathbf{p}_f)$. This is shown to have different statistics to the first correlation, and we derive a limit theorem for the average of the second correlation effect (details can be found in the Appendix). As it turns out, fluctuations of this average increase upon averaging, because when larger sets of data are considered there is a finite probability of finding an effect whose magnitude is $\sim n^2$ times larger than its typical value.

II. CROSS SECTION ASYMMETRY FOR A SPINLESS PARTICLE

Let us first study the simple case of the $\mathbf{p}_i \cdot \mathbf{p}_f$ correlation for scattering of a spinless particle. Here we consider the difference between forward and backward elastic scattering differential cross sections near threshold, due to interference of the *p*-wave resonance scattering with the background *s*-wave amplitude. For a spinless particle the scattering amplitude at low momenta is written using the Breit-Wigner formula as (see, e.g., [6])

$$f(\theta) = -A - \frac{g_p}{2k} \frac{\Gamma_p^{(n)}}{E - E_p + \frac{i}{2}\Gamma_p} \cos\theta, \tag{2}$$

where A is the *s*-wave scattering length, k and E are the wave number and energy of the projectile, and g_p, E_p, $\Gamma_p^{(n)}$, and Γ_p, are the statistical weight, energy, capture (or elastic) width, and total width of the *p* resonance, respectively. We assume that at energy E the *s*-wave background is nonresonant, and there is a *p*-wave resonance nearby, a condition which would favor larger asymmetries. This leads to expressions for the forward $(+)$ and backward $(-)$ scattering amplitudes

$$f^{\pm} = -A \mp \frac{g_p}{2k} \frac{\Gamma_p^{(n)}}{\varepsilon}, \tag{3}$$

where $\varepsilon = E - E_p$ is the distance to the *p*-wave resonance, and we assume that it is greater then the resonance width, i.e., $\varepsilon \gg \Gamma_p$.

The relevant observable is the asymmetry

$$x = \frac{(d\sigma/d\Omega)_+ - (d\sigma/d\Omega)_-}{(d\sigma/d\Omega)_+ + (d\sigma/d\Omega)_-}, \tag{4}$$

where $(d\sigma/d\Omega)_{\pm}$ are the forward and backward scattering cross sections. Substituting amplitudes (3) and taking into account that at low momenta the contribution of the *p* wave is much smaller than that of the *s* wave, we obtain

$$x = \frac{\Gamma_p^{(n)}}{\beta\varepsilon}, \tag{5}$$

where $\beta = Ak/g_p$. A typical value of this asymmetry is $g_p \Gamma_p^{(n)}/AkD$.

A. Statistical analysis

We would now like to obtain the probability density for the observable x. The capture width is proportional to the square of the capture amplitude. The capture amplitudes for complex compound nuclear states have a Gaussian distribution [7], hence, the widths $\Gamma_p^{(n)}$ are distributed according to the Porter-Thomas law

$$g(\gamma) = \frac{1}{\sqrt{2\pi\bar{\gamma}\gamma}} \exp\left(-\frac{\gamma}{2\bar{\gamma}}\right), \tag{6}$$

where $\gamma \equiv \Gamma_p^{(n)}$ for convenience and $\bar{\gamma}$ is the mean width $\bar{\gamma} = \int \gamma g(\gamma) d\gamma$.

For a given energy E the distance to the nearest *p* resonance in a compound nucleus is random. If the relative positions of the *p* resonances were uncorrelated it would be described by a Poissonian distribution

$$f_D(\varepsilon) = D^{-1} \exp\left(-\frac{2|\varepsilon|}{D}\right), \tag{7}$$

where D is the mean spacing between the *p* resonances and $\overline{|\varepsilon|} = D/2$. Correlations between the positions of compound states of the same symmetry, often referred to as level repulsion, modify the above distribution. These correlations are described by the random matrix theory [8], and can be approximated by the Wigner law. In this case the distance to the nearest *p* resonance has the following probability density [1]:

$$f_D(\varepsilon) = D^{-1} \exp\left(-\frac{\pi\varepsilon^2}{D^2}\right). \tag{8}$$

To avoid confusion we should stress that here ε is the distance to the nearest *p*-wave resonance and not the interval between the *p*-wave resonances. Therefore, Eq. (8) differs from the Wigner-Dyson distribution. The difference between Eqs. (7) and (8) is not very important for our consideration, as long as $f_D(\varepsilon)$ remains finite (and equal to D^{-1}) at $\varepsilon \to 0$ [1].

Using Eqs. (6) and (8) we calculate the distribution of the observable x as

$$f(x) = \int_0^\infty d\gamma \int_{-\infty}^\infty d\varepsilon f_D(\varepsilon) g(\gamma) \delta\left(x - \frac{\gamma}{\beta\varepsilon}\right)$$

$$= \frac{1}{\sqrt{2\pi x_0 |x|}} \int_0^\infty \sqrt{t} \exp\left(-\frac{|x|t}{2x_0} - \pi t^2\right) dt, \tag{9}$$

where $x_0 = |\bar{\gamma}/\beta D|$ represents some typical value of the effect. The integral in Eq. (9) can be given explicitly in terms of the parabolic cylinder functions $D_\nu(z)$,

$$f(x)=\frac{(2\pi)^{-3/4}\Gamma\left(\frac{3}{2}\right)}{\sqrt{2\pi x_0|x|}}\exp\left(\frac{x^2}{32\pi x_0}\right)D_{-3/2}\left(\frac{|x|}{x_0\sqrt{8\pi}}\right). \tag{10}$$

On the other hand, one can easily find the asymptotic behavior of the probability density at $|x|\gg x_0$ directly from Eq. (9):

$$f(x)\simeq x_0/x^2. \tag{11}$$

The CLT will not apply to distributions with this asymptotic form, since they do not have a finite variance: $\int x^2 f(x)dx \to\infty$.

B. Limit theorem for the first correlation

Suppose the forward-backward asymmetry is measured in a situation where a number of different nuclei with similar-sized cross sections are involved. The measurement will then yield some average asymmetry, and we want to find the probability distribution of it. Otherwise, one may just analyze the asymmetries measured separately for a number of nuclei. Let us then consider the average of n independent random variables x_i introduced above,

$$X=\frac{1}{n}\sum_{x=1}^{n} x_i. \tag{12}$$

In Ref. [1] a derivation of the limit theorem for distributions with asymptotic behavior (11) was presented. A general solution of the problem of limit distributions of sums of independent variables with an infinite variance for which $f(x)\propto|x|^{-\alpha-1}$, can be found in Ref. [3] [$\alpha>0$ to keep the total probability $\int f(x)dx$ finite]. A simple derivation of the limit theorem for such distributions is given in the Appendix.

The random variable x has a symmetric probability distribution, $f(-x)=f(x)$. In this case for $\alpha=1$ the limit distribution is obtained from Eqs. (A20) and (A22) with $a=0$ and $c=\pi x_0$. So, for $n\to\infty$ the probability density $F_n(X)$ approaches its limit form

$$F_n(X)=\frac{1}{\pi}\frac{X_c}{X^2+X_c^2}, \tag{13}$$

where $X_c=\pi x_0$ [1]. This is called the Cauchy distribution, and its main property is that it is independent of n, in particular, it does not become narrower as n increases. Therefore, fluctuations are not suppressed by averaging. Compare this with the standard central limit theorem, where the width decreases as $\sigma_n=\sigma_1/\sqrt{n}$.

The parameter X_c for our physical observable is given by $X_c=\pi\overline{\gamma}/\beta D=\pi g_p\overline{\Gamma_p^{(n)}}/AkD$. Throughout the derivation we considered the scattering length A as a constant. Indeed, if the energy E does not coincide with an s-wave resonance, the s-wave amplitude $-A$ represents the potential scattering amplitude. It does not vary strongly between isotopes, or nuclei of similar masses, because the nuclear potential does not vary much. The energy scale of its variation is MeV, similar to single-particle energy level spacing. Contrast this with the scale of the compound resonance spacings which are of the order of 10 eV. This difference in energy scales means that the compound resonance states can be treated statistically, while the scattering length is treated as a constant.

C. Influence of the resonance widths on the statistics of the average asymmetry

The above calculations have been based on the assumption that $\varepsilon\gg\Gamma_p$, so that the possible effects of the resonance widths have been neglected. This is justified, as long as the probability of finding a very small interval $\varepsilon\sim\Gamma_p$ is indeed small. However, it is easy to see the role of the width as we increase n. As we explained in the Introduction, the nonvanishing fluctuations of the average depends on having one value of the effect large, $x_i\sim nx_c$, where x_c is a typical value. Indeed, we can expect that if we make n measurements then at least one will have an energy spacing of the order $\varepsilon\sim D/n$, thus giving a large asymmetry (5). The energy denominator, however, can not be made arbitrarily small, and as smaller $E-E_p$ are considered, it will reach a limit $|\varepsilon+i\Gamma_p/2|\sim\Gamma_p$. Thus $D/n\sim\Gamma_p$ determines the largest values of n, beyond which the Cauchy distribution of the average effect begins to turn into a Gaussian one. Hence our statistical analysis is valid until $n\sim D/\Gamma_p$ (in heavy nonfissionable nuclei $D/\Gamma_p\sim 500$).

If we continue to take measurements after this and further increase the number of measurements n, the maximum value of the asymmetries will stabilize, being of the order $x_i\sim x_0 D/\Gamma_p=\pi\overline{\gamma}/\beta\Gamma_p$. Hence we will no longer have increasingly large values of the effect to continue the ''nonvanishing'' averaging. Thus, when $n\gg D/\Gamma_p$ the Gaussian statistics take over, the standard central limit theorem applies, and the usual $1/\sqrt{n}$ suppression of fluctuations takes place. Note that $x_0 D/\Gamma_p\gg x_0$ in fact determines the true finite, but large, variance of x. Beyond this value $f(x)$ decreases faster than x^{-2}, and it effectively determines the lower and upper limits in the variance integral $\int x^2 f(x)dx$.

III. SPIN ONE-HALF PARTICLE CORRELATIONS

Let us now consider scattering of low-energy spin-$\frac{1}{2}$ particles, looking at both the $\mathbf{p}_i\cdot\mathbf{p}_f$ and $\boldsymbol{\sigma}\cdot(\mathbf{p}_i\times\mathbf{p}_f)$ correlations. Again we assume that there is no nearby compound nucleus s-wave resonance and that the asymmetry is dominated by one nearby p-wave resonance. This is justified because the statistics of the average effect X for large n, $F_n(X)$, is determined by large values of the individual effects, i.e., by the asymptotic large-x behavior of the probability density $f(x)$, cf. Eq. (11) [9].

Consider scattering of the neutron, $s=\frac{1}{2}$ from a nucleus with spin I. The total angular momentum of the p-wave neutron is $\mathbf{j}=\mathbf{l}+\mathbf{s}$ and the total angular momentum of the compound resonance is $\mathbf{J}=\mathbf{I}+\mathbf{j}$. The amplitude of p-wave resonant scattering at arbitrary angle θ between the incoming and outgoing particle can be written as (see Ref. [10])

$$f_p = -\frac{1}{2k} \sum_{\substack{j,j_z,m \\ j',j'_z,m'}} C^{JJ_z}_{II_z j' j'_z} C^{j' j'_z}_{1m'(1/2)s'_z} \sqrt{4\pi} Y^*_{1m'}(\mathbf{n}'_k) \times \sqrt{\Gamma^{(n)}_{p_{j'}}(E)} \frac{1}{E - E_p + \frac{i}{2}\Gamma_p} C^{JJ_z}_{II_z j j_z} C^{j j_z}_{1m\frac{1}{2}s_z} \times \sqrt{4\pi} Y_{1m}(\mathbf{n}_k) \sqrt{\Gamma^{(n)}_{p_j}(E)}, \quad (14)$$

where I_z, $\mathbf{n}_k$, s_z, and I'_z, $\mathbf{n}'_k$, s'_z, describe the projection of the target spin, the direction of the neutron momentum, and the projection of the neutron spin in the initial and final states, respectively, $\Gamma^{(n)}_{p_j}$ is the capture width for the neutron with angular momentum j, the Y_{lm} are the angular wave functions, and $C^{JJ_z}_{II_z jj_z}$ are the Clebsch-Gordan coefficients.

Consider scattering of a neutron incident along the x direction off a spinless ($I=0$) nucleus. We quantize the neutron spin in the z direction and consider neutrons scattered in the xy plane. The s-wave scattering amplitude is simply $-A\delta_{s_z s'_z}$. Having in mind that we need to calculate interference terms between the s and p waves, we can write the p-wave scattering amplitude (14) in the following form:

$$f_p(\theta) = -\frac{1}{2k} \sum_m |C^{jj_z}_{1m(1/2)s_z}|^2 Y^*_{1m}(\mathbf{n}'_k) \times \frac{1}{E - E_p + \frac{i}{2}\Gamma_p} Y_{1m}(\mathbf{n}_k) 4\pi \Gamma^{(n)}_{p_j}(E). \quad (15)$$

A. First correlation

The resonance p-wave amplitude for forward and backward scattering (see Sec. II) with $j=\frac{1}{2}$ is

$$f^{\pm}_{p_{1/2}} = \mp\frac{1}{2k} \frac{\Gamma^{(n)}_{p_{1/2}}}{E - E_p + \frac{i}{2}\Gamma_p}, \quad (16)$$

which is similar to the spinless particle scattering [Eq. (2)], with the parameter $g_p=1$. Similarly for $j=\frac{3}{2}$ states the amplitude is

$$f^{\pm}_{p_{3/2}} = \mp\frac{2}{2k} \frac{\Gamma^{(n)}_{p_{3/2}}}{E - E_p + \frac{i}{2}\Gamma_p}, \quad (17)$$

which is similar to spinless particle scattering with $g_p=2$. This means that the statistics derived for the spinless particle $\mathbf{p}_i \cdot \mathbf{p}_f$ correlation in Sec. II are valid for the case where spin is included. In fact, since we do not know whether the nearest resonance is $p_{1/2}$ or $p_{3/2}$ we must combine the two distributions.

B. Second correlation

The second correlation $\boldsymbol{\sigma} \cdot (\mathbf{p}_i \times \mathbf{p}_f)$ between the direction of the spin relative to the scattering plane is, of course, specific to particles with a nonzero spin. To calculate the asymmetry of the cross section with respect to flipping the spin, we take the initial neutron momentum along the x-direction as before, and look at the difference between the scattering amplitude in the $+y$ direction, f^+, and that in the $-y$ direction, f^-. Equation (15) yields the p-resonance scattering amplitudes in the $+y$ and $-y$ directions for $j=\frac{1}{2}$

$$f^{\pm}_{p_{1/2}} = \pm\frac{i}{2k} \frac{\Gamma^{(n)}_{p_{1/2}}}{E - E_p + \frac{i}{2}\Gamma_p}, \quad (18)$$

and similarly for $j=\frac{3}{2}$ we obtain

$$f^{\pm}_{p_{3/2}} = \mp\frac{i}{2k} \frac{\Gamma^{(n)}_{p_{3/2}}}{E - E_p + \frac{i}{2}\Gamma_p}, \quad (19)$$

which differs from the $j=\frac{1}{2}$ case only by sign.

Thus the total scattering amplitude for the $\boldsymbol{\sigma} \cdot (\mathbf{p}_i \times \mathbf{p}_f)$ correlation is given by

$$f^{\pm} = -A \mp \frac{i\eta_p}{2k} \frac{\Gamma^{(n)}_p}{E - E_p + \frac{i}{2}\Gamma_p}, \quad (20)$$

where $\eta_p = -1$ for $j=\frac{1}{2}$ and $\eta_p = +1$ for $j=\frac{3}{2}$. Then, taking into account that the second term in Eq. (20), which represents the p-wave contribution, is much smaller than the first one, we obtain for the observable difference of the corresponding cross sections [see Eq. (4)]

$$x = \frac{\eta_p}{2kA} \frac{\Gamma_p \Gamma^{(n)}_p}{(E - E_p)^2 + \frac{1}{4}\Gamma_p^2}. \quad (21)$$

As we discussed above, the scattering length varies weakly. The same is true for the total width of the compound resonances Γ_p. Its fluctuations are small because it is dominated by the radiative width, given by the a sum of a large number of partial widths due to transitions into all lower-lying nuclear states. Introducing $\beta = 2kA/\eta_p\Gamma_p$, and taking into account that $\varepsilon = E - E_p$ is usually much larger than the resonance width, $|\varepsilon| \gg \Gamma_p$, we obtain for the asymmetry

$$x = \frac{\Gamma^{(n)}_p}{\beta\varepsilon^2}. \quad (22)$$

The typical size of this effect $\eta_p \Gamma^{(n)}_p \Gamma_p / AkD^2 = (\eta_p \Gamma^{(n)}_p / AkD)(\Gamma_p / D)$, is much smaller than the first correlation, by a factor of Γ_p/D. However, this observable has a ε^{-2} dependence on the distance to the nearest p resonance, while the first correlation was proportional to ε^{-1}. The ε^{-2} singularity emphasizes even stronger the possibility of small

denominators. Note also, that for a given scattering length A the sign of this interference effect is always the same. We will see that this leads to a very different statistics of the $\boldsymbol{\sigma}\cdot(\mathbf{p}_i\times\mathbf{p}_f)$ correlation.

1. Statistical analysis

Let us derive a probability distribution for the observable x given by Eq. (22). Similarly to Sec. II A we have

$$f(x)=\frac{1}{\sqrt{2\pi\bar{\gamma}}}\int_0^\infty\int_{-\infty}^\infty \exp\left(-\frac{\pi\varepsilon^2}{D^2}-\frac{\gamma}{2\bar{\gamma}}\right)\frac{d\varepsilon\, d\gamma}{\sqrt{\gamma}} \times\delta\left(x-\frac{\gamma}{\beta\varepsilon^2}\right), \tag{23}$$

where we again use γ for the capture width $\Gamma_p^{(n)}$, and the probability densities $g(\gamma)$ and $f_D(\varepsilon)$ are taken from Eqs. (6) and (8), respectively. Assuming that the scattering length is positive, hence, $\beta>0$, we calculate the above integral and obtain

$$f(x)=\frac{\sqrt{x_0}}{\sqrt{\pi x}(x+\pi x_0)},\quad x>0, \tag{24}$$

and $f(x)=0$ for $x<0$, where

$$x_0=\frac{2\bar{\gamma}}{\beta D^2} \tag{25}$$

characterizes typical values of the asymmetry (22). The asymptotic behavior of this probability density at $x\gg x_0$ is

$$f(x)\simeq\frac{\sqrt{x_0}}{\sqrt{\pi}|x|^{3/2}}. \tag{26}$$

The probability density $f(x)$ is normalized as $\int_0^\infty f(x)dx=1$. However, the corresponding mean value $\int f(x)xdx$ is infinite, and the integral for the variance $\int f(x)x^2dx$ diverges even faster than that for the first correlation [cf. Eq. (11)]. This signifies even larger fluctuations of the second correlation effect.

2. Limit theorem for the second correlation

The probability distribution of the second correlation (26) corresponds to $\alpha=\frac{1}{2}$ (see Appendix). Using the asymptotic parameters $c_1=0$ and $c_2=\sqrt{x_0/\pi}$ [compare Eqs. (26) and (A1)] we obtain $c=\sqrt{2x_0}$ and $\beta=1$ from Eqs. (A9) and (A10). In fact, it is possible to calculate the Fourier transform of $f(x)$ of Eq. (24) explicitly,

$$\tilde{f}(\omega)=e^{i\pi x_0\omega}\pi^{-1/2}\Gamma(1/2,i\pi x_0\omega) =e^{i\pi x_0\omega}[1-\sqrt{2x_0}(1\pm i)|\omega|^{1/2}+O(\omega^{3/2})], \tag{27}$$

where $\Gamma(\ldots)$ is the incomplete Γ-function, and $\pm$ corresponds to $\omega\gtrless 0$.

It follows now from Eqs. (A20) and (A24) that the limit distribution $F_n(X)$ of the average effect X is given by

$$F_n(X)=\sqrt{\frac{nx_0}{\pi}}\frac{e^{-nx_0/X}}{X^{3/2}}\quad (X>0). \tag{28}$$

This equation shows explicitly that as the number of effects included in the average increases, the distribution widens proportionally to n. Accordingly, the typical values of the average also grow as $X\sim nx_0$.

To understand this recall that the second asymmetry is inversely proportional to ε^2. Thus, when we consider the average of n such variables, one of them is likely to have $\epsilon\sim D/n$, which makes it n^2 times greater than the typical value x_0. This variable will dominate the average and give $X\sim nx_0$. More realistically, one must combine the distributions with different η_p remembering that the probability to find a close $p_{3/2}$ resonance is twice that of a $p_{1/2}$ resonance.

The role of resonance widths can be understood in the same way as it was done in Sec. II C. The limit statistics (28) is valid until $n\sim D/\Gamma_p$, i.e., when the characteristic value of the maximal effect that dominates the average X requires denominators as small as $\varepsilon\sim D/n\sim\Gamma$. When $n\gg D/\Gamma_p$ the usual statistics of the central limit theorem become valid, and we eventually have typical values of the average decreasing as $1/\sqrt{n}$.

IV. CONCLUSION

We have considered interference effects between p-wave resonance neutron scattering amplitude and background s-wave amplitude in compound nuclei and found that these effects do not obey the standard central limit theorem. That is, the probability density of the average effect over n measurements, $X=(1/n)\Sigma_{i=1}^n x_i$, does not tend to a Gaussian distribution with variance $\sigma_n^2=\sigma_1^2/n$ for large n. We have examined two effects, (i) the $\mathbf{p}_i\cdot\mathbf{p}_f$ correlation, which corresponds to the forward-backward asymmetry of the differential cross section, and (ii) the $\boldsymbol{\sigma}\cdot(\mathbf{p}_i\times\mathbf{p}_f)$ correlation, which describes the asymmetry of the scattering cross section with respect to flipping the spin relative to the scattering plane.

The first of these was found to have a distribution with asymptotic behavior $f(x)\propto 1/x^2$ for large x. In this case the limit theorem for the distribution of the average X tends to a Cauchy distribution $F_n(X)=X_c/[\pi(X^2+X_c^2)]$. This is independent of n, so the typical value of the average (X_c) does not decrease with increasing number of measurements. Physically this is understood by the following arguments. The asymmetry in question is inversely proportional to the spacing between the incident neutron energy and the energy of the closest p-wave resonance ($x\propto\varepsilon^{-1}$). If we have n measurements, we have a high probability that one of these spacings will be of the order $\varepsilon\sim D/n$ where D is the mean p level spacing. This will produce an asymmetry of the size $x\sim nx_0$ where x_0 is a typical value of the effect. Thus the typical average value is $X\sim x_0$, nonvanishing with increasing n.

The second correlation we considered produces a much

smaller effect than the first correlation. However, it has been found to have a ε^{-2} dependence, giving a distribution with asymptotic behavior $f(x)\propto x^{-3/2}$ for large x. This means that there is a higher probability to obtain relatively large values of x, compared to the first correlation. As a result, typical values for the average of n asymmetries actually increase with increasing number of measurements n as $X\sim nx_0$.

Above we assumed $\epsilon>\Gamma_p$ where ϵ is the distance to the resonance and Γ_p is the resonance width. When we consider the the influence of the resonance widths, it is found that they affect the distribution by limiting the size of the effect, x. Indeed, the minimum value for the denominator $|\varepsilon+i\Gamma_p/2|$ is given by $\sim\Gamma_p$, hence, the maximal possible effects are limited. We have found that our analysis of the statistics of the averages X is valid for $n<D/\Gamma_p$ (for heavy nonfissionable nuclei $D/\Gamma_p\sim 500$). For $n\gg D/\Gamma_p$ we expect the average to once again obey the standard CLT and vanish with increasing n.

APPENDIX: LIMIT THEOREM FOR PROBABILITY DISTRIBUTIONS WITH INFINITE VARIANCES

Consider a random variable whose probability density has the following asymptotic behavior:

$$f(x)=\begin{cases} c_1/|x|^{\alpha+1}, & x\to-\infty,\\ c_2/x^{\alpha+1}, & x\to+\infty, \end{cases} \tag{A1}$$

with $0<\alpha<2$, and is normalized in the usual way, $\int f(x)dx=1$. The existence of the mean, $\int xf(x)dx$, depends on whether α is greater or less than unity, but the variance integral $\int x^2 f(x)dx$ is infinite in both cases, and the standard central limit theorem is inapplicable.

To derive the limit statistics of the average $X=(1/n)\Sigma_{x=1}^{n}x_i$ of n independent random variables x_i, we use characteristic functions (or Fourier transforms)

$$\tilde f(\omega)=\int_{-\infty}^{\infty}e^{-i\omega x}f(x)dx. \tag{A2}$$

The Fourier transform of the probability density $F_n(X)$ of the average X is given by

$$\tilde F_n(\omega)=\int_{-\infty}^{\infty}e^{-i\omega X}dX\int\delta\left(X-\frac{1}{n}\sum_{i=1}^{n}x_i\right)\prod_{i=1}^{n}f(x_i)dx_i$$
$$=\prod_{i=1}^{n}\tilde f(\omega/n)=[\tilde f(\omega/n)]^n. \tag{A3}$$

Thus, the form of $F_n(\omega)$ for large n is related to that of $\tilde f(\omega)$ at small ω. This is in turn decided by the large-x asymptotic behavior of $f(x)$ given by Eq. (A1).

For $1<\alpha<2$ the integral $\int xf(x)dx$ converges and $\tilde f(\omega)$ can be written as

$$\tilde f(\omega)=1-i\omega\int xf(x)dx+\int(e^{-i\omega x}-1+i\omega x)f(x)dx. \tag{A4}$$

Let us consider the contribution of the interval from 0 to $+\infty$ to the last term above. Using the asymptotic form (A1) we present it as

$$\int_0^{\infty}(e^{-i\omega x}-1+i\omega x)\left[f(x)-\frac{c_2}{x^{\alpha+1}}\right]dx+c_2$$
$$\times\int_0^{\infty}(e^{-i\omega x}-1+i\omega x)\frac{dx}{x^{\alpha+1}}. \tag{A5}$$

If we assume that $f(x)$ approaches its asymptotic behavior sufficiently rapidly, e.g., $|f(x)-c_2/x^{\alpha+1}|\propto O(1/x^{\alpha+2})$, then the first integral above behaves as $O(\omega^2)$ at $\omega\to 0$. To calculate the second integral we turn the integration path into the complex plane by changing the variable $\omega x=-it$ (for $\omega>0$ the t is real positive), which gives

$$c_2e^{i(\pi\alpha/2)}\omega^{\alpha}\int_0^{\infty}t^{-\alpha-1}(e^{-t}-1+t)dt, \tag{A6}$$

where the integral is a representation of the Γ-function on a segment of the negative argument axis, $\Gamma(-\alpha)=-\Gamma(-\alpha+1)/\alpha$ [13].

Applying the same procedure to the integral over $(-\infty,0)$ in expression (A4), and turning the integration path into the complex plane by using $\omega x=it$ (for $\omega>0$ and real positive t), we obtain the expansion of $\tilde f(\omega)$ at small ω:

$$\tilde f(\omega)=1-i\omega a-(c_1e^{-i(\pi\alpha/2)}+c_2e^{i(\pi\alpha/2)})$$
$$\times\omega^{\alpha}\frac{\Gamma(\alpha-1)}{\alpha}+\cdots, \tag{A7}$$

where $a=\int xf(x)dx$ is the mean value. The expansion for negative ω can be obtained from the above by simply replacing ω with $|\omega|$ and complex-conjugating the exponential phase factors in the second term. Finally, at the same level of accuracy, we can rewrite expansion (A7) in the form valid for positive and negative small ω:

$$\tilde f(\omega)\simeq e^{-i\omega a}\left[1-c\left(1+i\beta\,\mathrm{sgn}(\omega)\tan\frac{\pi\alpha}{2}\right)|\omega|^{\alpha}\right], \tag{A8}$$

where

$$c\equiv(c_1+c_2)\frac{\Gamma(1-\alpha)}{\alpha}\cos\frac{\pi\alpha}{2},\quad c>0, \tag{A9}$$

$$\beta\equiv\frac{c_2-c_1}{c_2+c1},\quad -1\leqslant\beta\leqslant 1, \tag{A10}$$

and $\mathrm{sgn}(\omega)=\pm 1$ for $\omega>0$ and $\omega<0$, respectively. The parameters c and β [11] are determined by the asymptotic behavior (A1) of the probability density. The value of β depends on the asymmetry of the probability density $f(x)$.

The final form (A8) is very convenient. If we consider a random variable x_1 shifted with respect to x, $x_1=x-a$ (a is an arbitrary number here), its characteristic function would differ from that of x by a simple phase factor,

$\tilde{f}_1(\omega)=e^{i\omega a}\tilde{f}(\omega)$. On the other hand, the asymptotic behavior of the probability density, Eq. (A1) is not affected by this transformation. Therefore, the phase factor in Eq. (A8) can always be eliminated by this simple shift.

For $0<\alpha<1$ in Eq. (A1) we rewrite the Fourier transform as

$$\tilde{f}(\omega)=1+\int (e^{-i\omega x}-1)f(x)dx. \tag{A11}$$

The contribution of positive x to the above integral can be transformed into

$$\int_0^\infty (e^{-i\omega x}-1)\left[f(x)-\frac{c_2}{x^{\alpha+1}}\right]dx+c_2\int_0^\infty (e^{-i\omega x}-1)\frac{dx}{x^{\alpha+1}}. \tag{A12}$$

Provided the difference between $f(x)$ and $c_2/x^{\alpha+1}$ decreases as $x^{-\alpha-2}$ or faster, as $x\to+\infty$, the first integral can be expanded in powers of ω, with the leading term given by

$$i\omega\int_0^{+\infty} x\left[f(x)-\frac{c_2}{x^{\alpha+1}}\right]dx. \tag{A13}$$

The second integral in Eq. (A12) is transformed by variable substitution $\omega x=-it$ (for $\omega>0$) into [cf. Eq. (A6)]

$$c_2e^{i(\pi\alpha/2)}\omega^\alpha\int_0^\infty t^{-\alpha-1}(e^{-t}-1)dt, \tag{A14}$$

which again gives the Γ-function [13]. After we apply the same procedure to the negative-x part of the integral in Eq. (A12), the expansion of $\tilde{f}(\omega)$ at small ω is established in exactly the same form as that of Eq. (A7) (for $\omega>0$). However, for $0<\alpha<1$ the parameter a is no longer the mean value. Instead, it is given by

$$a=\int_{-\infty}^0 x\left[f(x)-\frac{c_1}{|x|^{\alpha+1}}\right]dx+\int_0^\infty x\left[f(x)-\frac{c_2}{x^{\alpha+1}}\right]dx. \tag{A15}$$

Also, the next term omitted in Eq. (A7) may now be greater than $O(\omega^2)$. Nevertheless, the small-ω behavior of the Fourier transform is still represented by Eq. (A8).

If $\alpha=1$ in Eq. (A1), the expansion of $\tilde{f}(\omega)$ also contains $\omega\ln\omega$ terms. In this case it can be presented as

$$\tilde{f}(\omega)\simeq 1-i\omega a-c|\omega|\left[1-i\frac{2}{\pi}\beta\,\mathrm{sgn}(\omega)\ln|\omega|\right], \tag{A16}$$

where $c=(\pi/2)(c_1+c_2)$, which can be obtained from Eq. (A9) at $\alpha\to 2$, β is given by Eq. (A10), and

$$a=(c_2-c_1)(1-\boldsymbol{C})+\int_{-\infty}^0 x\left[f(x)-\frac{c_1}{1+x^2}\right]dx+\int_0^\infty x\left[f(x)-\frac{c_2}{1+x^2}\right]dx, \tag{A17}$$

where $\boldsymbol{C}\approx 0.577$ is the Euler constant. Note that if the probability distribution is symmetric asymptotically, i.e., $c_1=c_2$, then $\beta=0$, and a in Eqs. (A15) and (A17) is the mean value calculated in the principal value sense. If the probability density is fully symmetric, $f(-x)=f(x)$, then $\tilde{f}(\omega)$ is real, $a=\beta=0$, and the behavior of the characteristic function at small ω is especially simple:

$$\tilde{f}(\omega)\simeq 1-c|\omega|^\alpha. \tag{A18}$$

After establishing the form of $\tilde{f}(\omega)$ at small ω, Eq. (A8) for $\alpha\neq 1$, we can proceed to derive the limit theorem, starting from Eq. (A3):

$$\tilde{F}_n(\omega)=e^{-i\omega a}\left[1-\frac{c\left(1+i\beta\,\mathrm{sgn}(\omega)\tan\frac{\pi\alpha}{2}\right)n^{1-\alpha}|\omega|^\alpha}{n}\right]^n \simeq e^{-i\omega a}\exp\left[-c\left(1+i\beta\,\mathrm{sgn}(\omega)\tan\frac{\pi\alpha}{2}\right)n^{1-\alpha}|\omega|^\alpha\right], \tag{A19}$$

for large n (this formula appears in the theorem by Khintchine and Lévy as a canonical representation of stable probability distributions, see Ref. [3]). Using the last expression in $F_n(X)=(1/2\pi)\int e^{i\omega X}F_n(\omega)d\omega$ we obtain the limit distribution in the following form:

$$F_n(X)=n^{(\alpha-1)/\alpha}c^{-1/\alpha}f_{\alpha\beta}[n^{(\alpha-1)/\alpha}c^{-1/\alpha}(X-a)], \tag{A20}$$

where

$$f_{\alpha\beta}(x)=\int_{-\infty}^\infty e^{i\omega x-|\omega|^\alpha}\exp\left[-i\beta\,\mathrm{sgn}(\omega)\tan\frac{\pi\alpha}{2}|\omega|^\alpha\right]\frac{d\omega}{2\pi} \tag{A21}$$

is a universal function of the two parameters, α and β, normalized to unity: $\int f_{\alpha\beta}(x)dx=1$. The results for $\alpha=1$ are obtained in a similar way, with a replaced by $a+c(2/\pi)\beta\ln n$ in Eq. (A20), and $f_{1\beta}(x)$ given by Eq. (A21), in which $\tan(\pi\alpha/2)$ is replaced with $-(2/\pi)\ln|\omega|$.

Equation (A20) shows that for $0<\alpha<1$ the limit distribution of the average widens with the increase of n, i.e., fluctuations of the average increase with the number of variables averaged. Since $n^{(\alpha-1)/\alpha}a\to 0$ for $n\to\infty$, the shift of the distribution (A20) by a is actually unimportant in this case and one can put $a=0$. For $\alpha=1$ the shape of the distribution $F_n(X)$ does not depend on n, i.e., fluctuations are neither enhanced nor suppressed by averaging. If $\beta\neq 0$ the whole distribution is gradually shifted proportionally to $\ln n$ into the direction determined by the sign of β. For $1<\alpha$

<2 the distribution of the average does become narrower with n, however the rate of suppression of fluctuations, $X \propto n^{-(\alpha-1)/\alpha}$ is slower than the standard CLT $n^{-1/2}$. Again, for symmetrically distributed x_i, the limit distribution $F_n(X)$ is even simpler, as $a=\beta=0$ in Eqs. (A20) and (A21).

There are a few cases where $f_{\alpha\beta}$ and, hence, $F_n(X)$ are known explicitly. For $\alpha=1$, $\beta=0$ $(c_1=c_2)$ Eq. (A21) gives the Cauchy law,

$$f_{1,0}(x)=\frac{1}{\pi}\frac{1}{1+x^2}, \tag{A22}$$

and $c=\pi c_1$. For $\alpha=1/2$, $\beta=0$ the limit function can be expressed in terms of the error function $\Phi(s)=2\pi^{-1/2}\int_0^s e^{-t^2}dt$ [12]:

$$f_{1/2,0}(x)=-\frac{1}{2\sqrt{\pi}|x|^{3/2}}\mathrm{Im}\left\{e^{-i(\pi/4)}e^{-i/4x}\left[1-\Phi\left(\frac{1}{2\sqrt{ix}}\right)\right]\right\}. \tag{A23}$$

For the same $\alpha=1/2$ in the maximally asymmetric case, $c_1=0$, $c_2>0$, i.e., $\beta=1$, which takes place if the random variables x_i are positive, one easily obtains the following simple answer [3]:

$$f^{1}_{\frac{1}{2},1}(x)=\begin{cases}0, & x<0,\\ (2\pi)^{-1/2}e^{-1/2x}x^{-3/2}, & x>0.\end{cases} \tag{A24}$$

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