SOR201 Examples 1

In Questions 1 and 2, all events belong to $\mathcal{F}$, an event field of subsets of the sample space $\mathcal{S}$; the probability function P satisfies the axioms given in the lecture notes (and can therefore be assumed to satisfy the addition law and the complementarity rule).

1. (i) Prove that $\quad \mathrm{P}(A \cap \overline{B}) = \mathrm{P}(A) - \mathrm{P}(A \cap B)$.

   (ii) Prove that $\quad \mathrm{P}(\overline{A} \cap \overline{B}) = 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A \cap B)$.

   (iii) Show that the probability of exactly one of the events A or B occurring is $\mathrm{P}(A) + \mathrm{P}(B) - 2\mathrm{P}(A \cap B)$.

   (iv) Prove that

$$\begin{aligned}
\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B) &= \mathrm{P}(A)\mathrm{P}(\overline{B}) - \mathrm{P}(A \cap \overline{B}) \\
&= \mathrm{P}(\overline{A})\mathrm{P}(B) - \mathrm{P}(\overline{A} \cap B) \\
&= \mathrm{P}(\overline{A \cup B}) - \mathrm{P}(\overline{A})\mathrm{P}(\overline{B}).
\end{aligned}$$

2. (i) Using the complementarity rule, prove that

$$\mathrm{P}(\bigcup_{i=1}^{n} A_i) = 1 - \mathrm{P}(\bigcap_{i=1}^{n} \overline{A_i})$$

   and

$$\mathrm{P}(\bigcap_{i=1}^{n} A_i) = 1 - \mathrm{P}(\bigcup_{i=1}^{n} \overline{A_i}).$$

   (ii) (a) Using the principle of induction, or otherwise, prove that

$$\mathrm{P}(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mathrm{P}(A_i).$$

   (b) Hence prove that

$$\mathrm{P}(\bigcap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} \mathrm{P}(A_i) - (n-1).$$

3. (i) Lift problem Suppose a lift has three occupants A, B and C and there are three possible floors (1, 2 and 3) on which they can get out. Assuming that each person acts independently of the others and that each person has an equally likely chance of getting off at each floor, calculate the probability that exactly one person will get out on each floor by

   (a) listing the total sample space and identifying the favourable outcomes;

   (b) enumerating the favourable outcomes and sample space.

   (ii) In the birthday problem for a group of n people (lecture notes §1.6.2), obtain an approximation to the probability p that there are no common birthdays by taking logarithms and using $\log_e(1-x) \approx -x$ for small positive x. Hence evaluate p approximately for $n = 30$.

/*continued overleaf*

4. Ten different pairs of shoes are in a cupboard. Four shoes are selected at random, without replacement. We want to calculate (to 2 decimal places) the probability p that there will be at least one pair among the selected shoes.

   (a) Let A_i be the event that pair i is chosen. Show that

   $$\mathrm{P}(A_i) = 3/95 \qquad \text{and} \qquad \mathrm{P}(A_i \cap A_j) = 1/4845, \quad i \neq j.$$

   Then use the generalized addition law to determine p.

   (b) Alternatively, find p by direct enumeration.

   [Hint: If *no* pair is selected, then we have selected one shoe from each of 4 different pairs.]

   (c) Extend your argument in (b) to find the probability that exactly one pair will be selected.

5. Each packet of a certain cereal contains a small plastic model of one of five different dinosaurs: a given packet is equally likely to contain any one of the five dinosaurs. Find the probability that someone buying six packets of the cereal will acquire models of their three favourite dinosaurs.

   [Hint: Let A_i be the event that the i^{th} favourite dinosaur is *not* acquired.]

Additional questions (NOT for handing in)

6. Prove by induction that

$$\sum_{i=1}^{n} \mathrm{P}(A_i) - \sum_{i<j} \mathrm{P}(A_i \cap A_j) \;\le\; \mathrm{P}(A_1 \cup \ldots \cup A_n).$$

[Hint: You may use the result in Question 2(ii)(a).]

7. Suppose we have a set of 6 cups and saucers, 2 yellow, 2 blue and 2 green. The cups are randomly set on the saucers. Calculate the probability that no cup is on a saucer of the same colour by *listing* all such outcomes (and *enumerating* the sample space).

8. (*Harder*) In the card matching problem discussed in lectures (§1.5.1), obtain an expression for the probability that exactly k matches occur. Give an approximation to this probability when N, the number of cards, is large.

   [Hint: First suppose that matches occur on a particular set of k cards. Using the result in lectures, write down the *probability* that there are no matches on any of the other $(N-k)$ cards, and deduce the *number of ways* in which this can occur. By taking account of the number of ways in which the k matching cards can be chosen, obtain the desired probability.]

9. Suppose that n people play a series of r games in which each game has one winner. Each player has an equal chance of winning a game and the outcome of any game is unaffected by the outcomes of the previous games. Find an expression for the probability that at least one player does not win a game in the series.

   [Hint : Let A_i be the event that player i does *not* win a game in the series. Explain why $\mathrm{P}(A_i) = \left(\frac{n-1}{n}\right)^r, \quad i = 1, ..., n$.]

10. (*Harder*) Suppose that n married couples are seated randomly along one side of a long table. Obtain an expression for the probability p that no husband sits next to his wife. Evaluate p in the case $n = 4$.

    [Hint: Use the same argument (suitably adapted) as in §1.6.3 of the lecture notes.]