SOR201 Examples 2

1. (i) Let the event $B \in \mathcal{F}$ with $\mathrm{P}(B) > 0$. Prove that the conditional probability function $\mathrm{P}(\cdot|B)$ satisfies the three conditions for a probability function.

   (ii) If $A_1, A_2, \ldots, A_n \in \mathcal{F}$ and $\mathrm{P}(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) > 0$, prove that

$$\mathrm{P}(A_1) > 0, \quad \mathrm{P}(A_1 \cap A_2) > 0, \ldots, \mathrm{P}(A_1 \cap \ldots \cap A_{n-2}) > 0.$$

   [Hint : What is the relationship between $A_1 \cap \ldots \cap A_{n-1}$ and $A_1 \cap \ldots \cap A_{n-2}$, and so on ?]

   Hence prove the (generalised) multiplication rule for conditional probabilities.

2. (i) State and prove Bayes' rule.

   (ii) A ball is in any one of n boxes. It is in the ith box with probability p_i. If the ball is in box i, a search of that box will uncover it with probability α_i. Show that the conditional probability that the ball is is box j, given that a search of box i did not uncover it, is

$$\begin{cases} \dfrac{p_j}{1-\alpha_i p_i}, & \text{if } j \neq i \\ \dfrac{(1-\alpha_i)p_i}{1-\alpha_i p_i}, & \text{if } j = i. \end{cases}$$

   (iii) One coin in 10,000,000 has two heads; one coin in 10,000,000 has two tails; the remaining coins are legitimate. If a coin, chosen at random, is tossed 10 times and comes up heads every time, what is the probability that it is two-headed? Suppose it falls heads n times in a row. How large must n be to make the odds approximately even that the coin is two-headed?

   (iv) Suppose a rare disease occurs by chance in 1 per 10,000 people. Suppose there is a diagnostic test with the following properties :

   if a person has the disease, the test will diagnose this correctly with probability 0.95;

   if a person does not have the disease, the test will diagnose this correctly with probability 0.995.

   If the test says that a person has the disease, calculate the probability that this is a correct diagnosis.

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3. (i) If A and B are independent events, prove that so are A and $\overline{B}$, $\overline{A}$ and B, and $\overline{A}$ and $\overline{B}$.
Discuss why any one of the four pairs being independent implies independence in each of the other three pairs.

(ii) (a) Consider the sample space

$$\{(a,b,c),(a,c,b),(b,a,c),(b,c,a),(c,a,b),(c,b,a),(a,a,a),(b,b,b),(c,c,c)\}.$$

Assign the probability of 1/9 to each sample point. Let A_i be the event that the ith place in a sample point is occupied by the letter a. Show that the events A_1, A_2, A_3 are pairwise independent but not completely independent.

(b) Consider the sample space $\mathcal{S} = \{E_1, E_2, E_3, E_4\}$ where

$$\mathrm{P}(E_1) = \sqrt{2}/2 - 1/4, \quad \mathrm{P}(E_2) = \mathrm{P}(E_4) = 1/4, \quad \mathrm{P}(E_3) = 3/4 - \sqrt{2}/2.$$

Let $A_1 = \{E_1, E_3\}$, $A_2 = \{E_2, E_3\}$, $A_3 = \{E_3, E_4\}$.
Show that $\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1).\mathrm{P}(A_2).\mathrm{P}(A_3)$ but that A_1, A_2, A_3 are not completely independent.

4. (i) A fair die is thrown n times. Let p_n be the probability of an even number of sixes in the n throws. (Zero is considered an even number.) Find a relationship between p_n and p_{n-1}, $(n \geq 2)$ and hence show that

$$p_n = \tfrac{1}{2}[1 + (\tfrac{2}{3})^n], \qquad n \geq 1.$$

(ii) In a series of independent trials a player has probabilities 1/3, 5/12 and 1/4 of scoring 0, 1 and 2 points respectively at each trial, the series continuing indefinitely. The scores are added. Let p_n be the probability of the player obtaining a total of exactly n points at some stage of play. Find the values of p_0, p_1 and p_2 and set up a difference equation for p_n, $(n \geq 3)$. Show that the solution is of the form

$$p_n = \tfrac{8}{11} + \tfrac{3}{11}(-\tfrac{3}{8})^n, \qquad n \geq 1.$$

Additional questions (NOT for handing in)

5. Let p_n denote the probability that in n tosses of a fair coin no run of three consecutive heads appears. Show that

$$\begin{aligned} p_n &= \tfrac{1}{2}p_{n-1} + \tfrac{1}{4}p_{n-2} + \tfrac{1}{8}p_{n-3} \\ p_0 &= p_1 = p_2 = 1. \end{aligned}$$

Find p_8.

(*Hint*: Condition on the occurrence of the first tail.)

6. Prove that if $A_1, A_2, ..., A_n$ are independent events, then

$$\mathrm{P}(A_1 \cup A_2 \cup ... \cup A_n) = 1 = \prod_{i=1}^{n} [1 - \mathrm{P}(A_i)].$$

7. A group of n players $\mathrm{P}_1, \mathrm{P}_2, ..., \mathrm{P}_n$ sit round a table, and each player in turn, starting with P_1, tosses a fair die. The die is passed around the table (making more than one circuit of the table if necessary) until a player throws a six and thereby wins the game. Show that the probability that P_i wins the game is

$$\frac{\frac{1}{6}\left(\frac{5}{6}\right)^{i-1}}{1 - \left(\frac{5}{6}\right)^n}, \quad i = 1, ..., n.$$

8. A random number N of fair dice is thrown, where

$$\mathrm{P}(N = n) = 2^{-n}, \quad n \geq 1.$$

Let S denote the sum of the scores on the dice. Find the probability that

   (a) $N = 2$, given $S = 3$;
   (b) $S = 3$, given N is odd.

9. Each of n urns contains a white balls and b black balls; the urns are numbered 1,2, $\ldots$, n. One randomly selected ball is transferred from the first urn into the second, then another from the second into the third, and so on. Finally a ball is drawn at random from the nth urn. Let

$$p_r = \mathrm{P}(\text{white ball drawn from the } r\text{th urn}).$$

Express p_r in terms of p_{r-1}, a and b for $r = 1, \ldots, n$.