SOR201 Examples 5

1. (i) Consider a sequence of independent trials each consisting of placing a ball at random into one of three cells. The system is in state k if exactly k cells are occupied. Show that this system is a Markov chain and find the transition probability matrix $\boldsymbol{P}$.
If initially all cells are empty, find the absolute probability distribution of X_2 i.e. find $\mathrm{P}(X_2 = k), (k = 0, 1, 2, 3)$ where $X_2 = k$ if the system is in state k after 2 trials.

   (ii) Let $Y_1, Y_2, \ldots$ be independent, identically distributed discrete random variables with probability distribution $\{\mathrm{P}(Y = k) = a_k, \quad k = 0, 1, 2, \ldots\}$. Let $X_n = Y_1 + Y_2 + \cdots + Y_n, \quad n = 1, 2, \ldots$ and $X_0 = 0$. Show that $\{X_n\}$ is a Markov chain with homogeneous transition probabilities. Find $\boldsymbol{P}$.

   (iii) *Ehrenfest Model for Diffusion*
M molecules are distributed in two urns A and B. At each time point a molecule is chosen at random and moved to the other urn. Let X_n denote the number of molecules in urn A immediately after the nth exchange. Show that $\{X_0, X_1, \ldots\}$ is a Markov chain with homogeneous transition probabilities. Find $\boldsymbol{P}$.

2. (i) Consider a Markov chain based on the two states 0 and 1 with transition probability matrix
$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

   (a) Calculate $\mathrm{P}(X_n = 1|X_{n-1} = 0), \mathrm{P}(X_m = 0|X_{m-2} = 1)$ and $\mathrm{P}(X_{r+3} = 1|X_r = 1)$.

   (b) Given that initially the process is equally likely to be in state 0 or state 1, calculate $\mathrm{P}(X_1 = 1), \mathrm{P}(X_2 = 1), \mathrm{P}(X_3 = 1)$.

   (c) Why can we use Markov's thereom to find an approximation to $\boldsymbol{P}^n$ when n is large? Calculate $\boldsymbol{P}^n$ as $n \to \infty$.

   (ii) Let $\{X_n\}$ be a Markov chain with state space 0,1,2, which is initially in state 0 and has transition probability matrix
$$\boldsymbol{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

   Find

   (a) $\mathrm{P}(X_0 = 0, X_1 = 1, X_2 = 1)$;

   (b) $\mathrm{P}(X_n = 1|X_{n-2} = 0)$;

   (c) the absolute probability distribution of X_2.

/continued overleaf

3. (i) Find the types and periods of the states of the Markov chains with the following transition probability matrices :

(a) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(ii) Find $\lim_{n\to\infty} \boldsymbol{P}^n$ where

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

4. Consider a sequence of independent Bernoulli trials, each with probability of success p. Let the process be in state $i, i = 0, 1, \ldots, (r-1)$, if i successes have been observed since the trials started, and in state r if <u>at least</u> r successes have been observed.

(a) Considering the two cases $r = 1$ and $r > 1$ separately, show that the process is a Markov chain with homogeneous transition probabilities and find $\boldsymbol{P}$.

(b) For $\underline{r = 1}$, show that, starting from state 0, absorption is certain and that the mean time to absorption is $1/p$.

(c) For $\underline{r > 1}$, derive a set of equations for $\{f_{ir}, i = 0, \ldots, (r-1)\}$, where f_{ir} denotes the probability of eventual absorption in state r, starting from state i. Also derive a set of equations for the mean times to absorption given that the process started in states $0, 1, \ldots, (r-1)$.

Additional questions (NOT for handing in)

5. N white balls and N red balls are randomly distributed into two cells (labelled 1 and 2) so that each cell contains N balls. At each subsequent step, one ball is selected at random from each cell and then placed in the other cell. Let X_n denote the number of white balls in cell 1 after n steps. Explain why $\{X_n, n \geq 0\}$ is a homogeneous Markov chain, and give its transition probability matrix. State the initial vector of absolute probabilities, and indicate how, for given N and n, you would calculate the vector of absolute probabilities after step n.

6. A homogeneous Markov chain $\{X_n : n = 0, 1, \ldots\}$ has states $\{0, 1, 2\}$ and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

At time $n = 0$, the system is equally likely to be in states 0, 1 or 2.

(a) Find $\mathrm{P}(X_2 = 1)$ and $\mathrm{P}(X_2 = 2)$.

(b) Explain why a limiting distribution $\boldsymbol{\pi}$ exists, and determine it.

7. Classify as transient or absorbing the states $\{0, 1, 2, 3, 4\}$ of the Markov chain with transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Let f_{ik} denote the probability that the system eventually enters the absorbing state k, given that it started in the transient state i. Write down (without proof) a set of equations for the probabilities $\{f_{ik}\}$, and hence determine f_{ik} for each i and k. What is the mean time to absorption from each transient state?

8. Classify the states $\{0, 1, 2, 3, 4, 5\}$ of a Markov chain with transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$