# SOR201 Examples 6

1. (i) Define the cumulative distribution function of a random variable $X$. Derive the following properties of $F$ :
   (a) $F(x) \leq F(y)$ when $x \leq y$;
   (b) $F(-\infty) = 0, \qquad F(+\infty) = 1$;
   (c) $\mathrm{P}(a < X \leq b) = F(b) - F(a)$.

   (ii) Suppose a device either fails instantaneously when turned on or fails eventually because of some continuous ageing process. Let the random variable $X$ denote the time to failure where $\mathrm{P}(X = 0) = p > 0$ and $\mathrm{P}(0 < X \leq x)$ has the same form as the cumulative distribution function of a negative exponential distribution.
   (a) Sketch the cumulative distribution function of $X$. Is it a continuous function ?
   (b) Find the cumulative distribution function of $X$.

   (iii) Suppose the probability density function of the random variable $X$ is symmetrical about the point $a$

$$\text{i.e. } f(a-y) = f(a+y), \quad y \geq 0.$$

   Show that

$$\int_{-\infty}^{a} f(x)dx = \int_{0}^{\infty} f(a-y)dy \qquad \text{and} \qquad \int_{a}^{\infty} f(x)dx = \int_{0}^{\infty} f(a+y)dy.$$

   Hence show that the median of $X$ is $a$. Using similar transformations, show that the mean of $X$ is also $a$.

2. (i) A continuous random variable $X$ has cumulative distribution function

$$F(x) = 1 - \exp(-x^2/2), \quad x \geq 0.$$

   Derive the probability density function of $X$ and then find the mean, variance, median and mode of the distribution. Give a rough sketch of the probability density function.
   **Hint** : Use the Gamma function in calculating the mean and variance.

   (ii) If the random variable $X$ has probability density function

$$f_X(x) = \begin{cases} kx^{p-1}/(1+x)^{p+q}, & x \geq 0; \quad p, q > 0 \\ 0, & \text{otherwise,} \end{cases}$$

   find the probability density function of $Y = 1/(1+X)$.

   (iii) Suppose the continuous random variable $X$ has the uniform distribution on [0,1]

$$\text{i.e. } f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   Find the probability density function of $Y = \log_e\{1/(1-X)\}$.

   (iv) Suppose $X$ is a continuous random variable with cumulative distribution function $F_X(x), -\infty < x < \infty$. Show that $Y = F_X(X)$ is uniformly distributed on [0,1].

3. (i) Find the probability density function of $Y = X^2$ when the probability density function of $X$ is given by
   (a) $f_X(x) = 2x\exp(-x^2), \quad 0 \leq x < \infty$;
   (b) $f_X(x) = (1+x)/2, \quad -1 \leq x \leq 1$;
   (c) $f_X(x) = 1/2, \quad -1/2 \leq x \leq 3/2$.

/continued overleaf

(ii) Suppose $Z \sim \mathrm{N}(0,1)$. Show that $V = Z^2$ has the $\chi^2$ distribution with 1 degree of freedom.

**Hints** : The p.d.f. of the $\chi^2$ distribution with $r$ degrees of freedom is

$f_V(v) = v^{r/2-1}\exp(-v/2)/\{2^{r/2}\Gamma(r/2)\}, \quad v \geq 0$; $r$ a positive integer: $\Gamma(1/2) = \sqrt{\pi}$.

(iii) Suppose the random variable $X$ is uniformly distributed on [0,1]

$$\text{i.e. } f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability density function and cumulative distribution function of $Y = X^\alpha$, where $\alpha$ may be positive or negative.
Sketch the probability density function of $Y$ when $\alpha = -1, \frac{1}{2}, 2$.

4. (i) Normal distribution Let $X \sim \mathrm{N}(\mu, \sigma^2)$.

(a) Prove that

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

**Hint** : Change the integral into a Gamma function integral.

(b) Prove that the random variable $W = a + bX$ is distributed $\mathrm{N}(a + b\mu, b^2\sigma^2)$.

(ii) Negative exponential distribution Let the probability density function of $X$ be

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0; \quad \lambda > 0.$$

(a) Using the Gamma function, find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(b) Show that the negative exponential distribution possesses the 'no memory' property i.e. for any $s, t > 0$, $\mathrm{P}(X > s + t | X > s) = \mathrm{P}(X > t)$.

(iii) Gamma distribution Let the probability density function of $X$ be

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1}\exp(-\lambda x)}{\Gamma(\alpha)}, \qquad x \geq 0; \quad \alpha, \lambda > 0.$$

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(b) Write down the probability density function of the $\chi^2$ distribution with $r$ degrees of freedom and its mean and variance.

(iv) Beta distribution Let $X \sim$ Beta $(a, b)$

$$\text{i.e. } f_X(x) = x^{a-1}(1-x)^{b-1}/B(a,b), \qquad 0 \leq x \leq 1; \quad a, b > 0.$$

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(b) Show that $Y = 1 - X$ is also Beta distributed.

(c) Suppose the random variable $W$ is defined over the finite interval [A,B], and is zero elsewhere, and the probability density function of $W$ has the same shape as that of $X$. What is the relationship between $W$ and $X$ ? Hence find the probability density function of $W$.

(v) Weibull distribution The cumulative distribution function of the two-parameter Weibull distribution is

$$F(x) = \begin{cases} 1 - \exp\{-(\frac{x}{b})^c\}, & x \geq 0; \quad b, c > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability density function of $X$ and hence its mean value.

(b) Calculate the survival function, hazard function and hazard rate function for this distribution. How does the hazard rate function behave for $c < 1, c = 1, c > 1$ ?