SOR201 Examples 7

1. (i) Suppose the continuous random variables (X, Y, Z) have joint

$$f(x, y, z) = Kxyz^2, \qquad 0 \le x, y \le 1, \quad 0 \le z \le 3.$$

 (a) Show that the constant $K = \frac{4}{9}$.
 (b) Find the marginal probability density function of Y and hence show that $\mathrm{E}(Y) = \frac{2}{3}$.
 (c) Show that the marginal joint probability density function of (X, Z) is

$$f_{X,Z}(x, z) = \tfrac{2}{9}xz^2, \qquad 0 \le x \le 1, \quad 0 \le z \le 3.$$

 (d) Find the conditional distribution of Y given $X = \frac{1}{2}, Z = 1$ and hence find $\mathrm{E}(Y|X = \frac{1}{2}, Z = 1)$.

 (ii) The joint probability density function of the random variables (X, Y) is

$$f(x, y) = K(1 - x)^\alpha y^\beta, \qquad 0 \le x \le 1, \quad 0 \le y \le x; \qquad \alpha, \beta > -1.$$

 (a) Sketch the area of the (x, y) plane where $f(x, y)$ is positive.
 (b) Explain why the marginal distribution of X has probability density function

$$g(x) = \int_0^x K(1 - x)^\alpha y^\beta \, dy, \qquad 0 \le x \le 1.$$

 Calculate $g(x)$ and hence find K in terms of α, β and a Beta function. Find the conditional probability density function $f(y|x)$ and hence calculate $\mathrm{E}(Y|x)$.
 (c) Similarly calculate the marginal probability density function of Y, $h(y)$, and the conditional probability density function $f(x|y)$. Indicate how to calculate $\mathrm{E}(X|y)$.

2. Suppose the random variables X and Y are independent and are Gamma distributed with parameters (α, λ) and (β, λ) respectively

$$\text{i.e. } f_X(x) = \frac{\lambda^\alpha x^{\alpha-1} \exp(-\lambda x)}{\Gamma(\alpha)}, \qquad x \ge 0; \quad \alpha, \lambda > 0,$$

with a similar expression for $f_Y(y)$.

 (a) By calculating the joint probability density function of $X+Y$ and X/Y, show that these random variables are independent and that $X+Y$ has the Gamma $(\alpha + \beta, \lambda)$ distribution. Find the probability density function of X/Y. Why are $X + Y$ and Y/X independent ? What is the probability density function of Y/X ?
 (b) Hence show that $X/(X+Y)$ and $X+Y$ are independent and that $X/(X+Y)$ is Beta distributed.
 (c) What are the corresponding results for the negative exponential distribution?
 (d) What are the corresponding results for the χ^2 distribution?

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3. (i) Let X and Y be independent continuous random variables with respective probability density functions $f_X(x)$, $-\infty < x < \infty$, and $f_Y(y)$, $-\infty < y < \infty$. Show that

(a) the probability density function of $U = XY$ is

$$\int_{-\infty}^{\infty} f_X(u/v) f_Y(v) |1/v| \, dv = \int_{-\infty}^{\infty} f_X(v) f_Y(u/v) |1/v| \, dv;$$

(b) the probability density function of $U = X/Y$ is

$$\int_{-\infty}^{\infty} f_X(uv) f_Y(v) |v| \, dv;$$

(c) the probability density function of $U = X + Y$ is

$$\int_{-\infty}^{\infty} f_X(v) f_Y(u-v) \, dv = \int_{-\infty}^{\infty} f_X(u-v) f_Y(v) \, dv.$$

(ii) Let X and Y be independent random variables, each distributed uniformly on [0,1] :

$$\text{i.e. } f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

with a similar expression for $f_Y(y)$.
Using part (i), or otherwise, find the probability density functions of

(a) XY (b) X/Y (c) $X + Y$.

4. (i) Suppose that X_1, X_2, X_3 are independent, identically distributed $\mathrm{N}(\mu, \sigma^2)$ random variables. Derive the joint probability density function of

$$U = X_1 - X_3, \qquad V = X_2 - X_3, \qquad W = X_1 + X_2 + X_3 - 3\mu$$

and hence obtain the joint probability density function of (U, V).

(ii) Let $Z_1, Z_2, \ldots, Z_n$ be independent N(0,1) random variables.

(a) Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be such that

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1, \qquad \sum_{i=1}^{n} a_i b_i = 0.$$

Define

$$Y_1 = \sum_{i=1}^{n} a_i Z_i, \quad Y_2 = \sum_{i=1}^{n} b_i Z_i, \quad W = \sum_{i=1}^{n} Z_i^2 - \{\sum_{i=1}^{n} a_i Z_i\}^2 - \{\sum_{i=1}^{n} b_i Z_i\}^2.$$

Explain why Y_1, Y_2 and W are independent random variables with $Y_1, Y_2 \sim \mathrm{N}(0,1)$ and $W \sim \chi^2(n-2)$.

(b) Let $X_1, X_2, \ldots, X_n$ be independent $\mathrm{N}(\mu, \sigma^2)$ random variables. Define the sample mean random variable $\overline{X}$ and the sample variance random variable S^2.
Quoting any relevant results for the random variables $Z_1, Z_2, \ldots, Z_n$, prove that $\overline{X}$ and S^2 are independent random variables and derive the distributions of $\overline{X}$, S^2 and $(\overline{X} - \mu)/\sqrt{S^2/n}$.