SOR201 Examples 8

1. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistic random variables associated with random samples of size n from the uniform distribution on [0,1]

$$\text{i.e. with p.d.f. } f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Show that the probability density function of $X_{(i)}$ is

$$f_{(i)}(x) = n!/\{(i-1)!(n-i)!\}x^{i-1}(1-x)^{n-i}, \quad 0 \le x \le 1$$

   and hence show that

$$\mathrm{E}(X_{(i)}) = p_i, \qquad \mathrm{Var}(X_{(i)}) = p_i q_i/(n+2)$$

   where $p_i = i/(n+1), \quad q_i = 1 - p_i$.

$$\underline{\textbf{Hints}} : \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b), \quad a, b > 0;$$

$$\Gamma(n+1) = n!, \quad n \ge 0.$$

   (b) Find the joint probability density function of $(X_{(i)}, X_{(j)}), \quad i < j$.

   (c) Find the joint probability density function of (R, W) where $R = X_{(n)} - X_{(1)}$ and $W = X_{(1)}$. Hence show that the probability density function of R is

$$f_R(r) = n(n-1)r^{n-2}(1-r), \qquad 0 \le r \le 1$$

   and hence calculate $\mathrm{E}(R)$ and $\mathrm{Var}(R)$.

2. (i) Show that the MGF of the negative exponential distribution with probability density function

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \ge 0; \quad \lambda > 0$$

   is given by

$$M_X(\theta) = \lambda/(\lambda - \theta), \quad \theta < \lambda.$$

   Hence find the mean and variance of the above distribution.

   (ii) (a) If $X_1, \ldots, X_n$ are independent random variables, each distributed with the probability density function in (i), show that $W = X_1 + X_2 + \cdots + X_n$ has the Gamma (n, λ) distribution i.e. shape or index parameter n and scale parameter λ.
   Hint : The Gamma (α, λ) distribution has probability density function

$$f_U(u) = \frac{\lambda^\alpha u^{\alpha-1}\exp(-\lambda u)}{\Gamma(\alpha)}, \qquad u \ge 0; \quad \alpha, \lambda > 0.$$

   (b) Using MGF, find the distribution of $\overline{X} = \sum_{i=1}^n X_i/n$.

   (c) If $U_1 \sim \mathrm{Gamma}(\alpha_1, \lambda)$, $U_2 \sim \mathrm{Gamma}(\alpha_2, \lambda)$ and U_1, U_2 are independent random variables, prove that

$$U_1 + U_2 \sim \mathrm{Gamma}(\alpha_1 + \alpha_2, \lambda).$$

/continued overleaf

3. (i) Let Y be distributed uniformly on [a,b]

$$\text{i.e. } f_Y(y) = \begin{cases} 1/(b-a), & a \leq y \leq b \\ 0, & \text{otherwise.} \end{cases}$$

Find the MGF of Y and hence calculate $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

(ii) Suppose that $X_1, \ldots, X_n$ are independent random variables, each distributed uniformly on [0,1].

(a) Show that X_i has MGF $(e^\theta - 1)/\theta$.

(b) Show that the MGF of $\overline{X} = \sum_{i=1}^n X_i/n$ is given by

$$M_{\overline{X}}(\theta) = [\{\exp(\theta/n) - 1\}/(\theta/n)]^n.$$

(c) By expanding $\log_e M_{\overline{X}}(\theta)$ in a power series in θ, show that

$$M_{\overline{X}}(\theta) = \exp\{\theta/2 + \theta^2/(24n) + O(1/n^2)\} \quad \text{for large } n.$$

Hence find an approximation to the distribution of $\overline{X}$ when n is large.

Hints : $\log_e(1+a) = a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \cdots$ provided $|a| < 1$.
MGF of $\mathrm{N}(\mu, \sigma^2)$ is $\exp(\mu\theta + \frac{1}{2}\sigma^2\theta^2)$.

4. (i) Suppose that $X_1, \ldots, X_n$ are independent random variables, where $X_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$, $(i = 1, \ldots, n)$. Using MGF, prove that

$$W = \sum_{i=1}^n a_i X_i \sim \mathrm{N}(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2).$$

In particular, if $X_i \sim \mathrm{N}(\mu, \sigma^2), (i = 1, \ldots, n)$, prove that the sample mean random variable $\overline{X} \sim \mathrm{N}(\mu, \sigma^2/n)$.

Hint : MGF of $\mathrm{N}(\mu, \sigma^2)$ is $\exp(\mu\theta + \frac{1}{2}\sigma^2\theta^2)$.

(ii) Define the bivariate MGF of the continuous random variables (X_1, X_2) with joint probability density function $f(x_1, x_2), \infty < x_1, x_2 < \infty$.
If $(X_1, X_2) \sim \mathrm{N}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2; \rho)$, their MGF is

$$M_{X_1,X_2}(\theta_1, \theta_2) = \exp\{\mu_1\theta_1 + \mu_2\theta_2 + \tfrac{1}{2}(\sigma_1^2\theta_1^2 + 2\rho\sigma_1\sigma_2\theta_1\theta_2 + \sigma_2^2\theta_2^2)\}.$$

(a) Calculate $\mathrm{E}(X_i), \mathrm{Var}(X_i), (i = 1, 2)$ and the correlation between X_1 and X_2.

(b) Find the distribution of $a_1X_1 + a_2X_2$.

(iii) Let $X_1, X_2, \ldots$ be a sequence of independent random variables, each distributed Poisson(1).

(a) What is the distribution of $S_n = X_1 + X_2 + \cdots + X_n$?

(b) What is the limit of the cumulative distribution function of $(S_n - n)/\sqrt{n}$ as $n \to \infty$?

(c) By finding the limit of $\mathrm{P}(S_n \leq n)$ as $n \to \infty$, prove that

$$e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) \to \frac{1}{2} \qquad \text{as } n \to \infty.$$