# SOR201 Examples 9

1. (i) Let $\{N(t), t \geq 0\}$ denote a Poisson process with rate $\lambda$.

   (a) Draw the transition rate diagram and hence find the transition rate matrix.

   (b) Write down an expression for $\mathrm{P}[N(t_2) - N(t_1) = n]$, where $t_1 < t_2$, explaining your answer.

   (c) Let the random variable $W_r$ denote the waiting time to the $r$th event. Then

   $$\mathrm{P}(W_r \leq w) = \mathrm{P}[N(w) \geq r] = 1 - \sum_{j=0}^{r-1} \frac{(\lambda w)^j \exp(-\lambda w)}{j!}.$$

   Find the probability density function of $W_r$ and state the probability distribution.

   (ii) Alternative derivation of the Poisson process Let $X_1, X_2, \ldots$ be independent random variables, each distributed negative exponential ($\lambda$). Define

   $$T_0 = 0 \qquad \text{and} \qquad T_n = \sum_{i=1}^{n} X_i, \quad n \geq 1$$

   $$\text{and} \qquad N(t) = \max\{n : T_n \leq t\}, \quad t \geq 0.$$

   (a) What is the distribution of $T_n, \quad n \geq 1$ ?

   (b) Explain why $N(t) \geq k$ occurs if and only if $T_k \leq t$.

   (c) Hence find an expression for $\mathrm{P}[N(t) \geq k]$ and show that $\{N(t), t \geq 0\}$ is a Poisson process.
   **Hint** : Using integration by parts and a recurrence relation, it may be shown that the cumulative distribution function of the random variable $U \sim$ Gamma $(m, \lambda)$, where $m$ is a positive integer, is such that $F_U(u) = \mathrm{P}(W \geq m)$, where $W \sim$ Poisson$(\lambda u)$.

2. (i) For the linear birth and death process with birth rate $\alpha$ equal to the death rate $\beta$, the PGF of $X(t)$ with the initial population size $X(0) = 1$ is

   $$G(s,t) = \frac{s - \alpha t(s-1)}{1 - \alpha t(s-1)}.$$

   Calculate the probabilities $\{p_n(t), n \geq 0\}$ and the mean population size at time $t$.
   **Hint** : Express the denominator of $G(s,t)$ in the form $C(1 - Ds)$.

   (ii) Pure birth process This is similar to the birth and death process discussed in the lectures except that there are no deaths i.e. $\beta_n = 0, n \geq 1$. Suppose the population is initially of size $a$. Derive an appropriate set of differential-difference equations for $\{p_n(t)\}$.

   (iii) Pure death process This is similar to the birth and death process discussed in the lectures except that there are no births i.e. $\alpha_n = 0, n \geq 0$. Suppose the population is initially of size $a$. Derive an appropriate set of differential-difference equations for $\{p_n(t)\}$.

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(iv) Immigration Consider the birth and death process discussed in the lectures and suppose that immigration can also occur, independently of births and deaths. The simplest assumption is that immigration occurs as a Poisson process with rate $\theta$, independent of population size

i.e. $\mathrm{P}[X(t+\delta t)=n+1|X(t)=n]=\theta\delta t+o(\delta t)$ for small $\delta t$ and $n\geq 0$.

Derive an appropriate set of differential-difference equations for $\{p_n(t)\}$.

3. (i) Consider a point process where type 1 and type 2 events can occur. Let the random variables $Y_1$ and $Y_2$ represent the times to a type 1 and type 2 event respectively. Suppose the random variables $Y_1, Y_2$ are independent and have negative exponential distributions with parameters $\lambda_1, \lambda_2$ respectively. Let $U = \min(Y_1, Y_2)$ i.e. the time to the first event. By calculating $\mathrm{P}(U > u)$, or otherwise, show that $U$ has the negative exponential distribution with parameter $\lambda_1+\lambda_2$. Generalize this result.

Suppose a queue has two servers, with independent service times distributed negative exponential($\lambda$). Using a simple diagram, discuss the departure rate.

(ii) Consider a queueing system with arrival rate $\lambda_n$ and departure rate $\mu_n$ when there are $n$ customers in the system. Suppose that the system is in equilibrium or steady state condition and that the equilibrium probabilities $\{\pi_n, n\geq 0\}$, where $\pi_n = \mathrm{P}(n$ customers in the system), satisfy the equations :

$$(\lambda_n+\mu_n)\pi_n=\lambda_{n-1}\pi_{n-1}+\mu_{n+1}\pi_{n+1}, \qquad n\geq 0;$$

$$\lambda_{-1}=\mu_0=0; \qquad \sum_{n=0}^{\infty}\pi_n=1.$$

Show that

$$\pi_n=\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\pi_0, \quad n\geq 1 \quad \text{and} \quad \pi_0=S^{-1} \quad \text{where} \quad S=1+\frac{\lambda_0}{\mu_1}+\frac{\lambda_0\lambda_1}{\mu_1\mu_2}+\cdots.$$

What condition must $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy ?

Find the probability distribution $\{\pi_n\}$ for the following cases :

(a) a queue with discouragement and constant serving rate

$$\text{e.g.} \qquad \lambda_n=\lambda/(n+1), \quad n\geq 0; \qquad \mu_n=\mu, \quad n\geq 1;$$

(b) M / M / s queue

$$\text{i.e.} \qquad \lambda_n=\lambda, \quad n\geq 0; \qquad \mu_n=\begin{cases} n\mu, & n\leq s \\ s\mu, & n>s; \end{cases}$$

(c) a queue with constant arrival rate and ample servers;

(d) a queue with constant serving rate and limited capacity

$$\text{e.g.} \qquad \lambda_n=\begin{cases} \lambda, & n<N \\ 0, & n\geq N. \end{cases} \qquad \mu_n=\mu, \quad n\geq 1$$