SOR201 Solutions to Examples 1

1. (i) $\mathrm{P}(A) = \mathrm{P}((A \cap B) \cup (A \cap \overline{B})) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap \overline{B})$ [m.e. events: axiom 3]
i.e. $\mathrm{P}(A \cap \overline{B}) = \mathrm{P}(A) - \mathrm{P}(A \cap B)$.

(ii) $\mathrm{P}(\overline{A} \cap \overline{B}) = \mathrm{P}(\overline{A \cup B}) = 1 - \mathrm{P}(A \cup B)$ [complementarity rule]
$= 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A \cap B)$. [addition law]

(iii) P(exactly one of A, B occurs) $= \mathrm{P}((A \cap \overline{B}) \cup (\overline{A} \cap B))$
$= \mathrm{P}(A \cap \overline{B}) + \mathrm{P}(\overline{A} \cap B)$ [m.e. events: axiom 3]
$= \mathrm{P}(A) - \mathrm{P}(A \cap B) + \mathrm{P}(B) - \mathrm{P}(B \cap A)$ [using part (i) result]
$= \mathrm{P}(A) + \mathrm{P}(B) - 2\mathrm{P}(A \cap B)$.

(iv) $\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B) = \mathrm{P}(A) - \mathrm{P}(A \cap \overline{B}) - \mathrm{P}(A)\mathrm{P}(B)$ [from part (i)]
$= \mathrm{P}(A)[1 - \mathrm{P}(B)] - \mathrm{P}(A \cap \overline{B})$
$= \mathrm{P}(A)\mathrm{P}(\overline{B}) - \mathrm{P}(A \cap \overline{B})$. [(complementarity rule]

The second result follows from symmetry: alternatively, using the first result,

$$\begin{aligned} \mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B) &= \mathrm{P}(B \cap A) - \mathrm{P}(B)\mathrm{P}(A) \\ &= \mathrm{P}(B)\mathrm{P}(\overline{A}) - \mathrm{P}(B \cap \overline{A}) \\ &= \mathrm{P}(\overline{A})\mathrm{P}(B) - \mathrm{P}(\overline{A} \cap B). \end{aligned}$$

To prove the third result:

$$\begin{aligned} \mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) - \mathrm{P}(A)\mathrm{P}(B) && \text{[addn. law]} \\ &= 1 - \mathrm{P}(\overline{A}) + 1 - \mathrm{P}(\overline{B}) - \mathrm{P}(A \cup B) \\ &\qquad -(1 - \mathrm{P}(\overline{A}))(1 - \mathrm{P}(\overline{B})) && \text{[complementarity]} \\ &= 1 - \mathrm{P}(\overline{A})\mathrm{P}(\overline{B}) - \mathrm{P}(A \cup B) && \text{[after cancellation]} \\ &= \mathrm{P}(\overline{A \cup B}) - \mathrm{P}(\overline{A})\mathrm{P}(\overline{B}). && \text{[complementarity]} \end{aligned}$$

2. (i)

$$\mathrm{P}(\bigcup_{i=1}^{n} A_i) = 1 - \mathrm{P}\left(\overline{\bigcup_{i=1}^{n} A_i}\right) \quad \text{[complementarity rule]}$$
$$= 1 - \mathrm{P}\left(\bigcap_{i=1}^{n} \overline{A}_i\right). \quad \text{[de Morgan, (1.8) in notes]}$$
$$\mathrm{P}(\bigcap_{i=1}^{n} A_i) = 1 - \mathrm{P}\left(\overline{\bigcap_{i=1}^{n} A_i}\right) \quad \text{[complementarity rule]}$$
$$= 1 - \mathrm{P}\left(\bigcup_{i=1}^{n} \overline{A}_i\right). \quad \text{[de Morgan, (1.9) in notes]}$$

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(ii) (a) Since

$$\begin{aligned} P(A_1 \cup A_2) &= P(A_1) + P(A_2) - P(A_1 \cap A_2) && \text{[addition law]} \\ &\leqslant P(A_1) + P(A_2), && \text{[by axiom } P(\cdot) \geqslant 0] \quad (1) \end{aligned}$$

the result holds for $n = 2$.
Now suppose that it holds for $n = m$ $(\geqslant 2)$, i.e. that

$$P(A_1 \cup ... \cup A_m) \leqslant \sum_{i=1}^{m} P(A_i). \qquad (2)$$

Then

$$\begin{aligned} P([A_1 \cup ... \cup A_m] \cup A_{m+1}) &\leqslant P(A_1 \cup ... \cup A_m) + P(A_{m+1}) && \text{[using (1)]} \\ &\leqslant \sum_{i=1}^{m} P(A_i) + P(A_{m+1}) && \text{[using (2)]} \\ &= \sum_{i=1}^{m+1} P(A_i), \end{aligned}$$

i.e. the result holds for $n = m + 1$. So by induction it holds for all $n \geqslant 2$.
[Note: this result concerns one side of the Bonferroni inequality ((1.17a) in lecture notes): for the other side, see Qn. 6 below.]

(b) We have that

$$\begin{aligned} P(\bigcap_{i=1}^{n} A_i) &= 1 - P(\bigcup_{i=1}^{n} \overline{A}_i) && \text{[from part (i)]} \\ &\geqslant 1 - \sum_{i=1}^{n} P(\overline{A}_i) && \text{[from part (ii)(a)]} \\ &= 1 - \sum_{i=1}^{n} \{1 - P(A_i)\} && \text{[complementarity]} \\ &= \sum_{i=1}^{n} P(A_i) - (n-1). \end{aligned}$$

3. (i) (a) **List total sample space and count favourable outcomes**
Outcome = (a, b, c), where a is the floor at which A gets out, etc.

A	B	C	A	B	C	A	B	C
1	1	1	2	1	1	3	1	1
1	1	2	2	1	2	3	1	2*
1	1	3	2	1	3*	3	1	3
1	2	1	2	2	1	3	2	1*
1	2	2	2	2	2	3	2	2
1	2	3*	2	2	3	3	2	3
1	3	1	2	3	1*	3	3	1
1	3	2*	2	3	2	3	2	2
1	3	3	2	3	3	3	2	2

Favourable outcomes are marked *
So P(one person gets out at each floor) $= 6/27 = 2/9$
(since the outcomes are equally likely).

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(b) **Enumerate sample space and favourable outcomes**
The favourable outcomes are the arrangements of 1,2,3: there are 3! = 6 such arrangements.
Total number of outcomes$= 3 \times 3 \times 3 = 27$.
So required probability = $6/27 = 2/9$.

(ii) From lecture notes (page 11):

$$p = (1-a)(1-2a)...(1-(n-1)a), \quad \text{where } a = 1/365.$$

So $\quad \log_e p = \sum_{r=1}^{n-1} \log_e(1-ra)$.

For small positive x, $\log_e(1-x) \approx -x$.

So $\quad \log_e p \approx -\sum_{r=1}^{n-1} ra = -a\sum_{r=1}^{n-1} r = -\frac{n(n-1)}{2}a = -\frac{n(n-1)}{730}$.

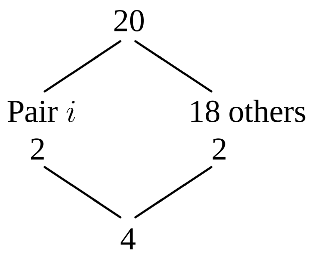
For $n = 30$:

$$\log_e p \approx -\frac{30 \times 29}{730} = -1.1918,$$

so $\quad p \approx 0.3037$.

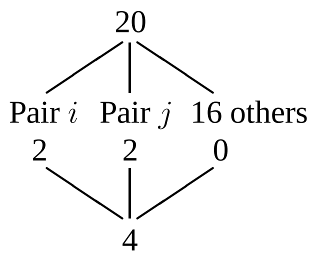
(The exact value is $p = 0.294$.)

4. (a) $\quad \mathrm{P}(A_i) = \frac{\binom{2}{2}\binom{18}{2}}{\binom{20}{4}} = \frac{4 \times 3}{20 \times 19} = \frac{3}{95}$.



By symmetry this result holds for $i = 1, ..., 10$.

For $i \neq j$: $\quad \mathrm{P}(A_i \cap A_j) = \frac{\binom{2}{2}\binom{2}{2}\binom{16}{0}}{\binom{20}{4}} = \frac{1}{4845}$.



Again by symmetry this result holds for $i, j = 1, ..., 10;\ i \neq j$.

Also $\quad \mathrm{P}(A_i \cap A_j \cap A_k) = 0, \quad i \neq j \neq k;$

$\quad \mathrm{P}(A_i \cap A_j \cap A_k \cap A_l) = 0; \quad i \neq j \neq k \neq l \quad$ etc.

Then

$$\begin{aligned} \mathrm{P}(\text{at least one pair}) &= \mathrm{P}(A_1 \cup A_2 \cup \cdots \cup A_{10}) \\ &= \sum_{i=1}^{10} \mathrm{P}(A_i) - \sum_{1 \leqslant i < j \leqslant 10} \mathrm{P}(A_i \cap A_j) \\ &= 10 \times \frac{3}{95} - \binom{10}{2} \times \frac{1}{4845} = \frac{99}{323} \approx \underline{0.31}. \end{aligned}$$

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(b) If no pair is selected, then the selection consists of *one* from each of 4 different pairs. The number of such selections is $\binom{10}{4}2^4$. The total number of possible (equally likely) selections is $\binom{20}{4}$. So

$$\mathrm{P}(\text{no pair}) = \frac{\binom{10}{4}2^4}{\binom{20}{4}} = \frac{224}{323}.$$

So $\mathrm{P}(\text{at least one pair}) = 1 - \frac{224}{323} = \frac{99}{323}$ (as in (a)).

(c) To find $\mathrm{P}(\textit{exactly}\text{ one pair})$: the one pair can be chosen in 10 ways, and the other 2 shoes in $\binom{9}{2}2^2$ ways. So

$$\mathrm{P}(\text{exactly one pair}) = \frac{10 \times \binom{9}{2}2^2}{\binom{20}{4}} = \frac{96}{323}.$$

5. Number the 3 favourite dinosaurs 1,2,3. Let

A_i: dinosaur i *not* found in purchase of 6 packets ($i = 1, 2, 3$).

The required probability is

$$\begin{aligned} \mathrm{P}(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3) &= 1 - \mathrm{P}(\overline{\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3}) \\ &= 1 - \mathrm{P}(A_1 \cup A_2 \cup A_3). \end{aligned}$$

[de Morgan: eqn. (1.9) of notes]

Now

$$\mathrm{P}(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} \mathrm{P}(A_i) - \sum_{1 \leqslant i < j \leqslant 3} \mathrm{P}(A_i \cap A_j) + \mathrm{P}(A_1 \cap A_2 \cap A_3),$$

where

$$\mathrm{P}(A_i) = \left(\frac{4}{5}\right)^6,$$

[by combinatorial argument: $\frac{4\times4\times4\times4\times4\times4}{5\times5\times5\times5\times5\times5}$ *or* using multiplication law for independent events]

$$\begin{aligned} \mathrm{P}(A_i \cap A_j) &= \left(\frac{3}{5}\right)^6, \quad i \neq j \\ \mathrm{P}(A_1 \cap A_2 \cap A_3) &= \left(\frac{2}{5}\right)^6. \end{aligned}$$

So the required probability is $1 - 3\left(\frac{4}{5}\right)^6 + 3\left(\frac{3}{5}\right)^6 - \left(\frac{2}{5}\right)^6.$

6. The result is true (as an equality) for $n = 2$ (by the addition law).

Assume that it is true for $n = m (\geqslant 2)$, i.e. that

$$\mathrm{P}(A_1 \cup \cdots \cup A_m) \geqslant \sum_{i=1}^{m} \mathrm{P}(A_i) - \sum_{1 \leqslant i < j \leqslant m} \mathrm{P}(A_i \cap A_j). \qquad (*)$$

Then

$$\mathrm{P}([A_1 \cup \cdots \cup A_m] \cup A_{m+1}) = \mathrm{P}(A_1 \cup \cdots \cup A_m) + \mathrm{P}(A_{m+1}) - \mathrm{P}([A_1 \cup \cdots \cup A_m] \cap A_{m+1}).$$

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The first term is developed using (*); we can write the third term as $\mathrm{P}(\bigcup_{i=1}^{m} (A_i \cup A_{m+1}))$ and then use the result in Qn. 2(ii)(a) above. Finally we obtain

$$\begin{aligned}\mathrm{P}(A_1 \cup \cdots \cup A_{m+1}) &\geqslant \sum_{i=1}^{m} \mathrm{P}(A_i) - \sum_{1 \leqslant i < j \leqslant m} \mathrm{P}(A_i \cap A_j) + \mathrm{P}(A_{m+1}) - \sum_{i=1}^{m} \mathrm{P}(A_i \cap A_{m+1}) \\ &= \sum_{i=1}^{m+1} \mathrm{P}(A_i) - \sum_{1 \leqslant i < j \leqslant m+1} \mathrm{P}(A_i \cap A_j),\end{aligned}$$

i.e. the result is true for $n = m + 1$. So by induction it is true for all $n \geqslant 2$.

7. Listing the outcomes in which no cup is placed on a saucer of the same colour:

Saucer:	Y	Y	B	B	G	G
Cup:	B	B	G	G	Y	Y
	G	G	Y	Y	B	B
	B	G	Y	G	B	Y
	B	G	Y	G	Y	B
	B	G	G	Y	B	Y
	B	G	G	Y	Y	B
	G	B	Y	G	B	Y
	G	B	Y	G	Y	B
	G	B	G	Y	B	Y
	G	B	G	Y	Y	B

There are 10 favourable outcomes altogether.

Total number of arrangements of cups on saucers is $\dfrac{6!}{2!2!2!} = 90$.

So the required probability is $\dfrac{10}{90} = \dfrac{1}{9}$.

(Solution of this problem by means of the generalized addition law is difficult.)

8. First consider matches on a particular set of k cards. The probability that there are *no* matches on any of the *other* $(n-k)$ cards is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-k}}{(n-k)!} \qquad (*)$$

(from result in Lectures applied to $(n-k)$ cards.) Now there are $(n-k)!$ ways of arranging the $(n-k)$ cards, so the number of arrangements which yield no matches is $(*) \times (n-k)!$. But also there are $\binom{n}{k}$ possible selections of the k matching cards. So the number of arrangements of all n cards in which there are exactly k matches is $\binom{n}{k} \times (n-k)! \times (*)$. The required probability is then obtained by dividing this by $n!$, the total number of arrangements, giving

$$\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-k}}{(n-k)!}\right] / k!$$

For large n this is approximately $e^{-1}/k!$. The values

$$e^{-1}/k!\,, \quad k = 0, 1, \ldots$$

are those associated with the Poisson distribution with mean 1.

[The above argument is more succinctly expressed using conditional probabilities (discussed in the next chapter of the lecture notes).]

9. Let A_i : player i does not win a game in the series.

Then

$$\begin{aligned}\text{P(at least one player does not win a game)} &= \mathrm{P}(A_1 \cup \cdots \cup A_n) \\ &= \sum_{i=1}^{n} \mathrm{P}(A_i) - \sum_{1 \leqslant i < j \leqslant n} \mathrm{P}(A_i \cap A_j) \\ &\quad + \cdots + (-1)^{n+1}\mathrm{P}(A_1 \cap \cdots \cap A_n).\end{aligned}$$

Now

$$\begin{aligned}\mathrm{P}(A_i) &= \mathrm{P}(\text{player } i \text{ doesn't win game } 1 \\ &\quad \text{\& player } i \text{ doesn't win game } 2 \\ &\quad \text{\& player } i \text{ doesn't win game } 3 \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad \text{\& player } i \text{ doesn't win game } n) \\ &= \left(\frac{n-1}{n}\right)^r, \qquad i = 1, \ldots, n.\end{aligned}$$

[*either* by combinatorial method: $\frac{(n-1).(n-1)....(n-1)}{n.n....n}$

or product of probabilities: $(\frac{n-1}{n})....(\frac{n-1}{n})$]

Similarly

$$\mathrm{P}(A_i \cap A_j) = \left(\frac{n-2}{n}\right)^r \qquad i \neq j$$

and so on. Finally $\mathrm{P}(A_1 \cap \cdots \cap A_n) = 0$.

So required probability is

$$n\left(\frac{n-1}{n}\right)^r - \binom{n}{2}\left(\frac{n-2}{n}\right)^r + \binom{n}{3}\left(\frac{n-3}{n}\right)^r - \cdots + (-1)^n\binom{n}{n-1}\left(\frac{1}{n}\right)^r.$$

10. Let A_i : couple i seated together. Required probability is then

$$\begin{aligned}1 - \mathrm{P}(A_1 \cup \cdots \cup A_n) &= 1 - \sum_{i=1}^{n} \mathrm{P}(A_i) + \sum_{1 \leqslant i_1 < i_2 \leqslant n} \mathrm{P}(A_{i_1} \cap A_{i_2}) \\ &\quad + \cdots + (-1)^n \mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_n).\end{aligned}$$

There are $(2n)!$ equally likely seatings. Consider a typical term $\mathrm{P}(A_{i_1} \cap \cdots \cap A_{i_k})$ in the k^{th} summation. Regard as k entities the k couples who are are seated together. There are $(2n-2k)$ other people, i.e. $(2n-k)$ entities in all. Number of (linear) arrangements is $(2n-k)!$ But also each couple can be seated in 2 ways. So the total number of seatings with k specified couples together is $2^k(2n-k)!$ So

$$\mathrm{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \frac{2^k(2n-k)!}{(2n)!}$$

and the required probability is then

$$1 - \binom{n}{1}\frac{2!(2n-1)!}{(2n)!} + \binom{n}{2}\frac{2!(2n-2)!}{(2n)!} - \cdots + (-1)^n\binom{n}{n}2^n\frac{n!}{(2n)!}.$$

For $n = 4$ this is

$$1 - \frac{4 \times 2}{8} + \frac{6 \times 4}{8 \times 7} - \frac{4 \times 8}{6 \times 7 \times 6} + \frac{16}{8 \times 7 \times 6 \times 5} = 1 - 1 + \frac{3}{7} - \frac{2}{21} + \frac{1}{105} = \frac{12}{35}.$$