SOR201 Solutions to Examples 2

1. (i) Let $\mathrm{Q}(A) = \mathrm{P}(A|B)$. Then

(a) for every event $A \in \mathcal{F}$,

$$\mathrm{Q}(A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \geqslant 0. \qquad \text{[by axiom 1]}$$

Also, since $A \cap B \subset B$, $\mathrm{P}(A \cap B) \leqslant \mathrm{P}(B)$, so $\mathrm{Q}(A) \leqslant 1$.

(b) $\mathrm{Q}(S) = \mathrm{P}(S|B) = \dfrac{\mathrm{P}(S \cap B)}{\mathrm{P}(B)} = \dfrac{\mathrm{P}(B)}{\mathrm{P}(B)} = 1.$

(c) Let $A_1, A_2, ...$ be mutually exclusive events in $\mathcal{F}$. Then

$$\begin{aligned}
\mathrm{Q}\left(\bigcup_i A_i\right) &= \frac{1}{\mathrm{P}(B)}\mathrm{P}\left((\bigcup_i A_i) \cap B\right) \\
&= \frac{1}{\mathrm{P}(B)}\mathrm{P}\left(\bigcup_i (A_i \cap B)\right) && \text{[distributive law]} \\
&= \frac{1}{\mathrm{P}(B)}\sum_i \mathrm{P}(A_i \cap B) && [\{A_i \cap B\} \text{ m.e.: axiom 3}] \\
&= \sum_i \mathrm{Q}(A_i).
\end{aligned}$$

So $\mathrm{Q}(\cdot) = \mathrm{P}(\cdot|B)$ satisfies the three probability axioms.

(ii) Since

$$\begin{aligned}
A_1 \cap A_2 \cap \cdots \cap A_{n-1} &\subset A_1 \cap A_2 \cap \cdots \cap A_{n-2} \\
&\subset A_1 \cap A_2 \cap \cdots \cap A_{n-3} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\subset A_1 \cap A_2 \\
&\subset A_1,
\end{aligned}$$

then

$$\begin{aligned}
0 &< \mathrm{P}(A_1 \cap \cdots \cap A_{n-1}) \\
&\leqslant \mathrm{P}(A_1 \cap \cdots \cap A_{n-2}) \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\leqslant \mathrm{P}(A_1 \cap A_2) \\
&\leqslant \mathrm{P}(A_1). \qquad (*)
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{P}(A_1 \cap \cdots \cap A_n) &= \mathrm{P}(A_n|A_1 \cap \cdots \cap A_{n-1})\mathrm{P}(A_1 \cap \cdots \cap A_{n-1}) \\
&= \mathrm{P}(A_n|A_1 \cap \cdots \cap A_{n-1})\mathrm{P}(A_{n-1}|A_1 \cap \cdots \cap A_{n-2}) \\
&\qquad \times \mathrm{P}(A_1 \cap \cdots \cap A_{n-2}) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \mathrm{P}(A_n|A_1 \cap \cdots \cap A_{n-1})\mathrm{P}(A_{n-1}|A_1 \cap \cdots \cap A_{n-2})\ldots.. \\
&\qquad \times \mathrm{P}(A_3|A_1 \cap A_2)\mathrm{P}(A_2|A_1)\mathrm{P}(A_1)
\end{aligned}$$

(by repeated application of the multiplication law $\mathrm{P}(A \cap B) = \mathrm{P}(A|B)\mathrm{P}(B)$ $(\mathrm{P}(B) > 0)$ and noting from $(*)$ that, since $\mathrm{P}(A_1 \cap \cdots \cap A_{n-1}) > 0$, all the conditioning events have probability > 0 as required).

2. (i) **Bayes' Rule** Let $H_1, H_2, \ldots, H_n$ be a set of mutually exclusive, exhaustive and possible events $\in \mathcal{F}$. For any event $A \in \mathcal{F}$ such that $\mathrm{P}(A) > 0$,

$$\mathrm{P}(H_k|A) = \frac{\mathrm{P}(A|H_k)\mathrm{P}(H_k)}{\sum_{j=1}^{n} \mathrm{P}(A|H_j)\mathrm{P}(H_j)}.$$

<u>Proof</u>

$$\begin{aligned}
\mathrm{P}(A) = \mathrm{P}(A \cap \mathcal{S}) &= \mathrm{P}\left(A \cap (\bigcup_{j=1}^{n} H_j)\right) \\
&= \mathrm{P}\left(\bigcup_{j=1}^{n} (A \cap H_j)\right) && \text{[distributive law]} \\
&= \sum_{j=1}^{n} \mathrm{P}(A \cap H_j) && [\{A \cap H_j\} \text{ m.e.: axiom 3}] \\
&= \sum_{j=1}^{n} \mathrm{P}(A|H_j)\mathrm{P}(H_j). && \text{[multiplication law] } (*)
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{P}(H_k|A) &= \frac{\mathrm{P}(H_k \cap A)}{\mathrm{P}(A)} && \text{[by definition]} \\
&= \frac{\mathrm{P}(A|H_k)\mathrm{P}(H_k)}{\sum_{j=1}^{n} \mathrm{P}(A|H_j)\mathrm{P}(H_j)} && \text{[using multiplication law and } (*)]
\end{aligned}$$

as required.

(ii) Let

A_i: search of box i does not uncover the ball

H_j: ball is in box j $(j = 1, ..., n)$.

Then we have

$$\begin{aligned}
\mathrm{P}(H_j) &= p_j \\
\mathrm{P}(A_i|H_j) &= \begin{cases} 1 - \alpha_i, & \text{if } j = i \\ 1, & \text{if } j \neq i \end{cases}
\end{aligned}$$

So by Bayes' Rule

$$\mathrm{P}(H_j|A_i) = \frac{\mathrm{P}(A_i|H_j)\mathrm{P}(H_j)}{\sum_{l=1}^{n} \mathrm{P}(A_i|H_l)\mathrm{P}(H_l)}.$$

The denominator is $(1 - \alpha_i)p_i + \sum_{l=1, l\neq i}^{n} 1 \times p_l = \sum_{l=1}^{n} p_l - \alpha_i p_i = 1 - \alpha_i p_i$.

So

$$\mathrm{P}(H_j|A_i) = \begin{cases} \dfrac{(1 - \alpha_i)p_i}{1 - \alpha_i p_i}, & \text{if } j = i \\ \dfrac{p_j}{1 - \alpha_i p_i}, & \text{if } j \neq i. \end{cases}$$

/continued overleaf

(iii) Define events as follows:

L: legitimate coin chosen
$H2$: 2-headed coin chosen
$T2$: 2-tailed coin chosen
nH: n heads in succession.

Then

$$\begin{aligned} \mathrm{P}(H2) &= 10^{-7} = \mathrm{P}(T2) \\ \mathrm{P}(L) &= 1 - 2\times 10^{-7} \\ \mathrm{P}(10H|H2) &= 1, \quad \mathrm{P}(10H|T2) = 0, \quad \mathrm{P}(10H|L) = 2^{-10}. \end{aligned}$$

So, using Bayes' Rule,

$$\begin{aligned} \mathrm{P}(H2|10H) &= \frac{\mathrm{P}(10H|H2)\mathrm{P}(H2)}{\mathrm{P}(10H|H2)\mathrm{P}(H2) + \mathrm{P}(10H|T2)\mathrm{P}(T2) + \mathrm{P}(10H|L)(\mathrm{P}(L)} \\ &= 10^{-7}/\{10^{-7} + 2^{-10}(1 - 2\times 10^{-7})\}. \end{aligned}$$

Since $2^{10} \approx 10^3$,

$$\mathrm{P}(H2|10H) \approx 1/\{1 + 10^{-3}\times 10^7\} \approx 10^{-4}.$$

Generalising to nH we have

$$\mathrm{P}(H2|nH) = \frac{\mathrm{P}(nH|H2)\mathrm{P}(H2)}{\mathrm{P}(nH|H2)\mathrm{P}(H2) + \mathrm{P}(nH|T2)\mathrm{P}(T2) + \mathrm{P}(nH|L)(\mathrm{P}(L)}$$

where

$$\mathrm{P}(nH|H2) = 1, \quad \mathrm{P}(nH|T2) = 0, \quad \mathrm{P}(nH|L) = 2^{-n},$$

so that

$$\mathrm{P}(H2|nH) = 10^{-7}/\{10^{-7} + 2^{-n}(1 - 2\times 10^{-7})\}.$$

For approximately even odds that the chosen coin is 2-headed, we require

$$\mathrm{P}(H2|nH) \approx \frac{1}{2},$$

i.e. we have to solve

$$\begin{aligned} 10^{-7} + 2^{-n}(1 - 2\times 10^{-7}) &\approx 2\times 10^{-7} \\ \text{or} \quad 2^{-n}(10^7 - 2) &\approx 1 \\ \text{or} \quad 2^n &\approx 10^7 \end{aligned}$$

for n. The solution of $2^x = 10^7$ is $x = 23.25$, so

$$n = 23 \quad \text{or} \quad 24.$$

(iv) Define events as follows:

D: a certain person has the disease
T^+: test diagnoses that the person has the disease
T^-: test diagnose that the person does not have the disease.

We have

$$\begin{aligned} \mathrm{P}(T^+|D) &= 0.95 \quad \Rightarrow \quad \mathrm{P}(T^-|D) = 0.05 \\ \mathrm{P}(T^-|\overline{D}) &= 0.995 \quad \Rightarrow \quad \mathrm{P}(T^+|\overline{D}) = 0.005. \end{aligned}$$

Then, by Bayes' Rule, the required probability is

$$\begin{aligned} \mathrm{P}(D|T^+) &= \frac{\mathrm{P}(T^+|D)\mathrm{P}(D)}{\mathrm{P}(T^+|D)\mathrm{P}(D) + \mathrm{P}(T^+|\overline{D})\mathrm{P}(\overline{D})} \\ &= \frac{0.95\times 0.0001}{(0.95\times 0.0001) + (0.005\times 0.9999)} = 0.019. \end{aligned}$$

/continued overleaf

[Although, with $\mathrm{P}(T^+|D) = 0.95,\ \mathrm{P}(T^-|\overline{D}) = 0.995$, the test appears at first sight to be a good one, the predictive positive probability (see below) is very low because of the very low prevalence probability $\mathrm{P}(D) = 0.0001$. Compare, for example, the case where $\mathrm{P}(D) = 0.01$: then we find that $\mathrm{P}(D|T^+) \approx 0.66$ – very much better.

[*Further notes*: The validity of a test is measured by its

	sensitivity	$\mathrm{P}(T^+|D) \times 100\%$
and	**specificity**	$\mathrm{P}(T^-|\overline{D}) \times 100\%$

– both of which should be high.

For a *patient*, however, the important measures are the

	predictive positive probability	$\mathrm{P}(D|T^+)$
and	**predictive negative probability**	$\mathrm{P}(\overline{D}|T^-)$,

which depend upon the **prevalence rate** $\mathrm{P}(D)$.]

3. (i) $A = A \cap \mathcal{S} = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$. [union of 2 m.e. events]

So

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(A \cap B) + \mathrm{P}(A \cap \overline{B}), && \text{[axiom 3]} \\ \text{i.e.} \quad \mathrm{P}(A \cap \overline{B}) &= \mathrm{P}(A) - \mathrm{P}(A \cap B) \\ &= \mathrm{P}(A) - \mathrm{P}(A)\mathrm{P}(B) && \text{[independence]} \\ &= \mathrm{P}(A)[1 - \mathrm{P}(B)] = \mathrm{P}(A)\mathrm{P}(\overline{B}). && \text{[complementarity]} \end{aligned}$$

So A and $\overline{B}$ are independent events.

To prove that $\overline{A}$ and B are independent, reverse the symbols A and B in the above proof.

Also

$$\begin{aligned} \mathrm{P}(\overline{A} \cap \overline{B}) = \mathrm{P}(\overline{A \cup B}) &= 1 - \mathrm{P}(A \cup B) && \text{[complementarity]} \\ &= 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A \cap B) && \text{[addition law]} \\ &= 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A)\mathrm{P}(B) && \text{[independence]} \\ &= \{1 - \mathrm{P}(A)\}\{1 - \mathrm{P}(B)\} \\ &= \mathrm{P}(\overline{A})\mathrm{P}(\overline{B}). && \text{[complementarity]} \end{aligned}$$

So $\overline{A}$ and $\overline{B}$ are independent events.

We have therefore proved that

independence of A, B $\Rightarrow$ independence of $A, \overline{B}$; of $\overline{A}, B$; & of $\overline{A}, \overline{B}$.

So

independence of $A, \overline{B}$ $\Rightarrow$ independence of $A, \overline{\overline{B}} = B$; of $\overline{A}, \overline{B}$; & of $\overline{A}, \overline{\overline{B}} = B$.

independence of $\overline{A}, B$ $\Rightarrow$ independence of $\overline{A}, \overline{B}$; of $\overline{\overline{A}} = A, B$; & of $\overline{\overline{A}} = A, \overline{B}$.

independence of $\overline{A}, \overline{B}$ $\Rightarrow$ independence of $\overline{A}, \overline{\overline{B}} = B$; of $\overline{\overline{A}} = A, \overline{B}$;
& of $\overline{\overline{A}} = A, \overline{\overline{B}} = B$.

(ii) (a) $\mathcal{S} = \{(a,b,c), (a,c,b), (b,a,c), (b,c,a), (c,a,b), (c,b,a), (a,a,a), (b,b,b), (c,c,c)\}$.
$\mathrm{P}(A_1) = \frac{3}{9} = \frac{1}{3};\ \mathrm{P}(A_2) = \frac{3}{9} = \frac{1}{3};\ \mathrm{P}(A_3) = \frac{3}{9} = \frac{1}{3}$.

$$\begin{aligned} \mathrm{P}(A_1 \cap A_2) &= \tfrac{1}{9} = \mathrm{P}(A_1)\mathrm{P}(A_2) \\ \mathrm{P}(A_1 \cap A_3) &= \tfrac{1}{9} = \mathrm{P}(A_1)\mathrm{P}(A_3) \\ \mathrm{P}(A_2 \cap A_3) &= \tfrac{1}{9} = \mathrm{P}(A_2)\mathrm{P}(A_3). \end{aligned}$$

But

$$\mathrm{P}(A_1 \cap A_2 \cap A_3) = \tfrac{1}{9} \neq \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3) \quad \left(= \tfrac{1}{27}\right).$$

So the events A_1, A_2, A_3 are *pairwise* independent but not *completely* independent.

/continued overleaf

(b) $\mathrm{P}(E_1) = \frac{\sqrt{2}}{2} - \frac{1}{4};\ \mathrm{P}(E_2) = \mathrm{P}(E_4) = \frac{1}{4};\ \mathrm{P}(E_3) = \frac{3}{4} - \frac{\sqrt{2}}{2}.$
So

$$\begin{aligned}\mathrm{P}(A_1) &= \mathrm{P}(E_1) + \mathrm{P}(E_3) = \tfrac{1}{2};\\ \mathrm{P}(A_2) &= \mathrm{P}(E_2) + \mathrm{P}(E_3) = 1 - \tfrac{\sqrt{2}}{2};\\ \mathrm{P}(A_3) &= 1 - \tfrac{\sqrt{2}}{2}.\end{aligned}$$

Now

$$\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(E_3) = \tfrac{3}{4} - \tfrac{\sqrt{2}}{2}$$

and

$$\mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3) = \tfrac{1}{2}\left(1 - \tfrac{\sqrt{2}}{2}\right)^2 = \tfrac{1}{2}(1-\sqrt{2}+\tfrac{1}{2}) = \tfrac{3}{4}-\tfrac{\sqrt{2}}{2} = \mathrm{P}(A_1 \cap A_2 \cap A_3).$$

But

$$\mathrm{P}(A_1 \cap A_2) = \mathrm{P}(E_3) = \tfrac{3}{4} - \tfrac{\sqrt{2}}{2} \neq \mathrm{P}(A_1)\mathrm{P}(A_2) \quad \left(= \tfrac{1}{2} - \tfrac{\sqrt{2}}{4}\right).$$

So the events A_1, A_2, A_3 are not completely independent.

4. (i) Let
E_n: even number of sixes in n throws
S: six on first throw

Then conditioning on the result of the first throw:

$$\begin{aligned}\mathrm{P}(E_n) &= \mathrm{P}(E_n|S)\mathrm{P}(S) + \mathrm{P}(E_n|\overline{S})\mathrm{P}(\overline{S})\\ &= \mathrm{P}(\overline{E}_{n-1})\mathrm{P}(S) + \mathrm{P}(E_{n-1})\mathrm{P}(\overline{S}),\end{aligned}$$

i.e.

$$\begin{aligned}p_n &= \tfrac{1}{6}(1 - p_{n-1}) + \tfrac{5}{6}p_{n-1}\\ &= \tfrac{2}{3}p_{n-1} + \tfrac{1}{6}, \quad n \geqslant 2. \qquad (1)\end{aligned}$$

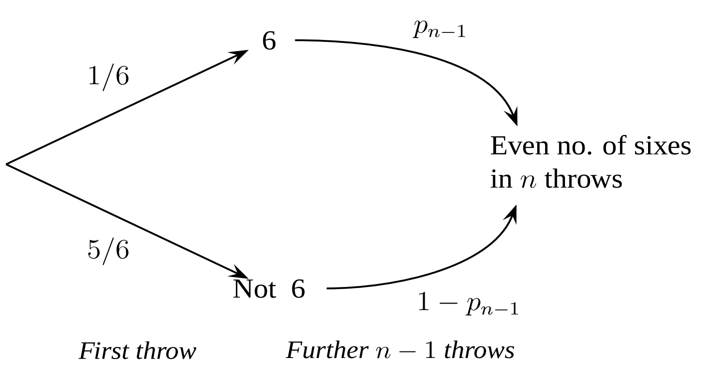
Also

$$p_1 = \mathrm{P}(\text{not six on first throw}) = \tfrac{5}{6}. \qquad (2)$$

Notes:

(a) This is 'first step analysis': alternatively we could have used 'last step analysis'.

(b) With experience, we can appeal to the diagram below and write down (1) directly (cf. solution to part (ii)).



To show that

$$p_n = \tfrac{1}{2}\left[1 + \left(\tfrac{2}{3}\right)^n\right], \quad n \geqslant 1 \tag{3}$$

we use induction. (3) is true for $n = 1$. Suppose it is true for $n = m$, where $m \geqslant 1$. Then (1) gives

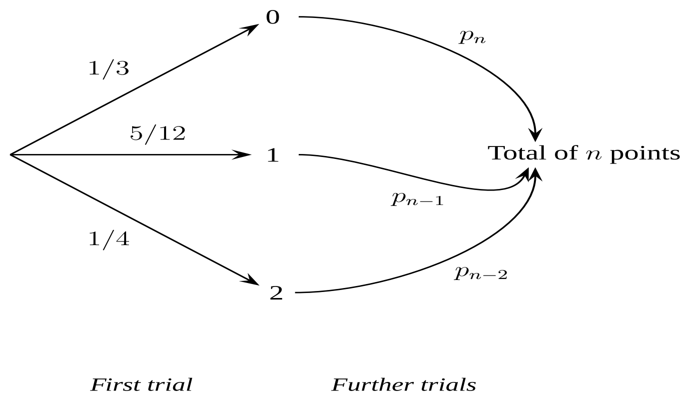
$$p_{m+1} = \tfrac{2}{3} \times \tfrac{1}{2}\left[1 + \left(\tfrac{2}{3}\right)^m\right] + \tfrac{1}{6} = \tfrac{1}{2}\left[1 + \left(\tfrac{2}{3}\right)^{m+1}\right],$$

i.e. it is true for $n = m + 1$. So by induction it is true for all $n \geqslant 1$.

(ii) Here

$$p_n = \mathrm{P}(\text{player obtains a total of exactly } n \text{ points at } \textit{some} \text{ stage of play}).$$

For $n \geqslant 3$, we can decompose the event of interest by again conditioning on the result of the first trial, as shown:



Hence

$$p_n = \tfrac{1}{3}p_n + \tfrac{5}{12}p_{n-1} + \tfrac{1}{4}p_{n-2}, \quad n \geqslant 3,$$

i.e.

$$p_n = \tfrac{5}{8}p_{n-1} + \tfrac{3}{8}p_{n-2}, \quad n \geqslant 3. \tag{1}$$

To start the recursion, we need p_1 and p_2. Now the event 'total of 1 at some stage' comprises the mutually exclusive events

$$\{1,\ 01,\ 001,\ 0001,\ ...\}.$$

So

$$p_1 = \tfrac{5}{12} + \tfrac{1}{3}\tfrac{5}{12} + \left(\tfrac{1}{3}\right)^2\tfrac{5}{12} + \cdots = \tfrac{5}{12} \times \tfrac{1}{1-\frac{1}{3}} = \tfrac{5}{8}.$$

Similarly, the event 'total of 2 at some stage' comprises the mutually exclusive events

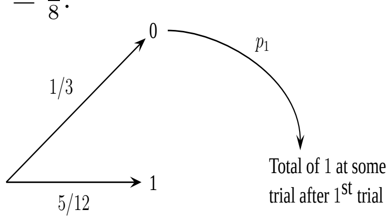
$$\{2,\ 02,\ 002,\ ...;r \text{ zeros and one '1', followed by a '1', } r \geqslant 0\}.$$

Hence

$$\begin{aligned} p_2 &= \sum_{r=0}^{\infty}\left(\tfrac{1}{4}\right)\left(\tfrac{1}{3}\right)^r + \sum_{r=0}^{\infty}\binom{r+1}{1}\left(\tfrac{5}{12}\right)\left(\tfrac{1}{3}\right)^r \times \tfrac{5}{12} \\ &= \tfrac{1}{4} \times \tfrac{1}{1-\frac{1}{3}} + \tfrac{1}{\left(1-\frac{1}{3}\right)^2}\left(\tfrac{5}{12}\right)^2 = \tfrac{49}{64}. \end{aligned}$$

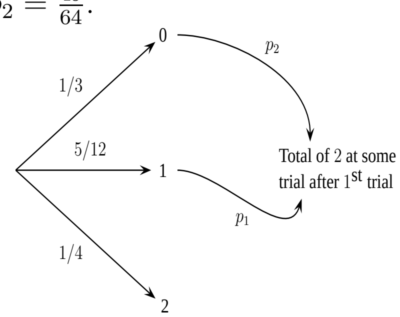
Alternatively we can use suitable diagrams:

(a) $p_1 = \frac{1}{3}p_1 + \frac{5}{12}$, giving $p_1 = \frac{5}{8}$.



(b) $p_2 = \frac{1}{3}p_2 + \frac{5}{12}p_1 + \frac{1}{4}$

which (using $p_1 = \frac{5}{8}$) gives $p_2 = \frac{49}{64}$.



To show that

$$p_n = \frac{8}{11} + \frac{3}{11}\left(-\frac{3}{8}\right)^n, \quad n \geqslant 1 \tag{2}$$

we first note that it is true for $n = 1, n = 2$, since it gives

$$\begin{aligned} p_1 &= \frac{8}{11} + \frac{3}{11}\left(-\frac{3}{8}\right) &= \frac{5}{8} \quad \surd \\ p_2 &= \frac{8}{11} + \frac{3}{11}\left(-\frac{3}{8}\right)^2 &= \frac{49}{64}. \quad \surd \end{aligned}$$

Now assume that it is true for $n = m - 2$ and $n = m - 1 \quad (m \geqslant 3)$. Then from (1):

$$\begin{aligned} p_m &= \frac{5}{8}\left\{\frac{8}{11} + \frac{3}{11}\left(-\frac{3}{8}\right)^{m-1}\right\} + \frac{3}{8}\left\{\frac{8}{11} + \frac{3}{11}\left(-\frac{3}{8}\right)^{m-2}\right\} \\ &= \frac{8}{11} + \frac{3}{11}\left(-\frac{3}{8}\right)^m \end{aligned}$$

i.e it is true for $n = m$. Hence by induction it is true for all $n \geqslant 1$.

5. Let

A: *no* run of 3 consecutive heads in n tosses

T_i: first tail occurs on the i^{th} toss.

Then

$$p_n = \mathrm{P}(A) = \mathrm{P}(A|T_1)\mathrm{P}(T_1) + \mathrm{P}(A|T_2)\mathrm{P}(T_2) + \mathrm{P}(A|T_3)\mathrm{P}(T_3)$$

$$[\mathrm{P}(A|T_i) = 0 \quad \text{for} \quad i = 4, ..., n]$$

But $\mathrm{P}(T_1) = \frac{1}{2}$, $\mathrm{P}(T_2) = \mathrm{P}(HT) = \left(\frac{1}{2}\right)^2$, $\mathrm{P}(T_3) = \mathrm{P}(HHT) = \left(\frac{1}{2}\right)^3$,

and $\mathrm{P}(A|T_i) = p_{n-i}, \quad i = 1, 2, 3.$

So

$$p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-2} + \frac{1}{8}p_{n-3}, \quad n \geqslant 3$$

To get three successive heads, we need at least 3 tosses, so clearly

$$p_1 = p_2 = p_3 = 1.$$

/continued overleaf

Then

$$\begin{aligned}
p_3 &= \tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8} && = \tfrac{7}{8}\\
p_4 &= (\tfrac{1}{2}\times\tfrac{7}{8})+(\tfrac{1}{4}\times 1)+(\tfrac{1}{8}\times 1) && = \tfrac{13}{16}\\
p_5 &= (\tfrac{1}{2}\times\tfrac{13}{16})+(\tfrac{1}{4}\times\tfrac{7}{8})+(\tfrac{1}{8}\times 1) && = \tfrac{3}{4}\\
p_6 &= (\tfrac{1}{2}\times\tfrac{3}{4})+(\tfrac{1}{4}\times\tfrac{13}{16})+(\tfrac{1}{8}\times\tfrac{7}{8}) && = \tfrac{11}{16}\\
p_7 &= (\tfrac{1}{2}\times\tfrac{11}{16})+(\tfrac{1}{4}\times\tfrac{3}{4})+(\tfrac{1}{8}\times\tfrac{13}{16}) && = \tfrac{81}{128}\\
p_8 &= (\tfrac{1}{2}\times\tfrac{81}{128})+(\tfrac{1}{4}\times\tfrac{11}{16})+(\tfrac{1}{8}\times\tfrac{3}{4}) && = \boldsymbol{\tfrac{149}{256}}
\end{aligned}$$

6. We have

$$\begin{aligned}
\mathrm{P}(A_1\cup A_2\cup\cdots\cup A_n) &= \mathrm{P}(\overline{\overline{A}_1\cap \overline{A}_2\cap\cdots\cap \overline{A}_n})\\
&= 1-\mathrm{P}(\overline{A}_1\cap\overline{A}_2\cap\cdots\cap\overline{A}_n) && \text{[complementarity]}\\
&= 1-\mathrm{P}(\overline{A}_1)\mathrm{P}(\overline{A}_2)\cdots\mathrm{P}(\overline{A}_n) && \text{[independence]}\\
&= 1-\prod_{i=1}^{n}[1-\mathrm{P}(A_i)]. && \text{[complementarity]}
\end{aligned}$$

7. Write $\overline{6}$ to mean 'not a 6', and let

A_{ik} : die makes k circuits (with all n players throwing $\overline{6}$ on each turn), then players $1, ..., i-1$ throw $\overline{6}$ and player i throws 6 (to win).

From independence it follows that

$$\mathrm{P}(A_{ik}) = \left[\left(\tfrac{5}{6}\right)^n\right]^k \left(\tfrac{5}{6}\right)^{i-1}\tfrac{1}{6}.$$

Then

$$\begin{aligned}
\mathrm{P}(\text{player } i \text{ wins}) &= \mathrm{P}(A_{i0}\cup A_{i1}\cup A_{i2}\cup\cdots)\\
&= \sum_{k=0}^{\infty}\mathrm{P}(A_{ik}) && \text{[m.e. events]}\\
&= \sum_{k=0}^{\infty}\left[\left(\tfrac{5}{6}\right)^n\right]^k \left(\tfrac{5}{6}\right)^{i-1}\tfrac{1}{6}\\
&= \frac{1}{1-\left(\frac{5}{6}\right)^n}\left(\tfrac{5}{6}\right)^{i-1}\tfrac{1}{6}. && \text{[geometric series]}
\end{aligned}$$

8. (a) By Bayes' Rule:

$$\mathrm{P}(N=2|S=3) = \frac{\mathrm{P}(S=3|N=2)\mathrm{P}(N=2)}{\mathrm{P}(S=3)}$$

$$\text{where } \mathrm{P}(S=3) = \sum_{k=1}^{\infty}\mathrm{P}(S=3|N=k)\mathrm{P}(N=k).$$

But

$$\begin{aligned}
\mathrm{P}(S=3|N=1) &= \tfrac{1}{6}\\
\mathrm{P}(S=3|N=2) &= \mathrm{P}(\{(1,2),(2,1)\}) = \tfrac{2}{36} = \tfrac{1}{18}\\
\mathrm{P}(S=3|N=3) &= \mathrm{P}((1,1,1)) = \tfrac{1}{6^3} = \tfrac{1}{216}\\
\mathrm{P}(S-3|N=k) &= 0 \quad \text{for } k \geqslant 4.
\end{aligned}$$

So

$$\mathrm{P}(S=3) = (\tfrac{1}{6}\times\tfrac{1}{2})+(\tfrac{1}{18}\times\tfrac{1}{4})+(\tfrac{1}{216}\times\tfrac{1}{8}) = \tfrac{169}{36\times 48}$$

and

$$\mathrm{P}(N=2|S=3) = \tfrac{1}{18}\times\tfrac{1}{4}\times\tfrac{36\times 48}{169} = \tfrac{24}{169}.$$

/continued overleaf

(b) $\mathrm{P}(S=3|N \text{ odd}) = \dfrac{\mathrm{P}((S=3)\cap(N \text{ odd}))}{\mathrm{P}(N \text{ odd})}$.

Now

$$\begin{aligned}
\mathrm{P}(N \text{ odd}) &= \mathrm{P}(N=1)+\mathrm{P}(N=3)+\cdots \\
&= \tfrac{1}{2}+(\tfrac{1}{2})^3+\cdots \\
&= \tfrac{1}{2}(1+\tfrac{1}{4}+(\tfrac{1}{4})^2+\cdots) \\
&= \tfrac{1}{2}\times\tfrac{1}{1-\frac{1}{4}} \\
&= \tfrac{2}{3},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}((S=3)\cap(N \text{ odd})) &= \sum_{k=1}^{\infty}\mathrm{P}((S=3)\cap(N \text{ odd})|N=k)\mathrm{P}(N=k) \\
&= \mathrm{P}(S=3|N=1)\mathrm{P}(N=1)+\mathrm{P}(S=3|N=3)\mathrm{P}(N=3) \\
&= (\tfrac{1}{6}\times\tfrac{1}{2})+(\tfrac{1}{216}\times\tfrac{1}{8}) \\
&= \tfrac{145}{36\times 48}
\end{aligned}$$

[Alternatively:

$$\begin{aligned}
\mathrm{P}((S=3)\cap(N \text{ odd})) &= \mathrm{P}(N \text{ odd}|S=3)\mathrm{P}(S=3) \\
&= [1-\mathrm{P}(N \text{ even}|S=3)]\mathrm{P}(S=3) \\
&= 1-\mathrm{P}(N=2|S=3)]\mathrm{P}(S=3) \\
&= [1-\tfrac{24}{169}]\times\tfrac{169}{36\times 48} \qquad \text{[using results in (a)]} \\
&= \tfrac{145}{36\times 48}. \qquad]
\end{aligned}$$

So

$$\mathrm{P}(S=3|N \text{ odd}) = \tfrac{145}{36\times 48}\times\tfrac{3}{2} = \mathbf{\tfrac{145}{1152}}.$$

9. Let

W_r: white ball drawn from urn r

B_r: black ball drawn from urn r

Conditioning on the result of the draw from urn $r-1$:

$$\begin{aligned}
p_r = \mathrm{P}(W_r) &= \mathrm{P}(W_r|W_{r-1})\mathrm{P}(W_{r-1})+\mathrm{P}(W_r|B_{r-1})\mathrm{P}(B_{r-1}) \\
&= \left(\frac{a+1}{a+b+1}\right)p_{r-1}+\left(\frac{a}{a+b+1}\right)(1-p_{r-1}) \\
&= \frac{1}{a+b+1}p_{r-1}+\frac{a}{a+b+1}, \qquad r=2,...,n
\end{aligned}$$

with $p_1 = \dfrac{a}{a+b}$.