SOR201 Solutions to Examples 3

1. (a) An outcome is an unordered sample $\{i_1, ..., i_n\}$, a subset of $\{1, ..., N\}$, where the i_j's are all different. The random variable X can take the values $n, n+1, ..., N$.
If $X = x$, the other numbers selected have values in the range $1, ..., (x-1)$.
If $X = x$, $(n-1)$ numbers have been selected from the set $\{1, ..., (x-1)\}$ and 0 numbers from the set $\{(x+1), ..., N\}$. This can be done in $\binom{x-1}{n-1}$ ways. The total number of possible selections is $\binom{N}{n}$. So

$$\mathrm{P}(X = x) = \binom{x-1}{n-1}\Big/\binom{N}{n}, \qquad x = n, ..., N.$$

(b) If the largest number in the sample is $\leqslant y$, then *all* numbers in the sample are $\leqslant y$, and vice versa; also $y \geqslant n$ and y need not be an integer.
Consider y to be one of the values $n, n+1, ..., N$. Then

$$\begin{aligned}
\mathrm{P}(X \leqslant y) &= \mathrm{P}(\text{all numbers in sample } \leqslant y) \\
&= \frac{\text{No. of ways of selecting } n \text{ numbers from } \{1, ..., y\}, \text{without replacement}}{\text{No. of ways of selecting } n \text{ numbers from } \{1, ..., N\}, \text{without replacement}} \\
&= \binom{y}{n}\Big/\binom{N}{n}, \qquad y = n, n+1, ..., N.
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{P}(X = x) &= \mathrm{P}(X \leqslant x) - \mathrm{P}(X \leqslant x-1) \\
&= \left[\binom{x}{n} - \binom{x-1}{n}\right]\Big/\binom{N}{n} \\
&= \binom{x-1}{n-1}\Big/\binom{N}{n}, \qquad x = n, ..., N.
\end{aligned}$$

(using the identity $\binom{y}{r-1} + \binom{y}{r} = \binom{y+1}{r}$).

2. (i) (a) $Y = X^2$ or $X = \pm\sqrt{Y}$.
So

$$\begin{aligned}
\mathrm{P}(Y = y) &= \mathrm{P}(X = \sqrt{y}) + \mathrm{P}(X = -\sqrt{y}) \\
&= p(\sqrt{y}) + p(-\sqrt{y}), \qquad y = 1, 4, 9, ... \\
\text{and} \quad \mathrm{P}(Y = 0) &= p(0).
\end{aligned}$$

(b) $W = |X| = \begin{cases} X, & \text{when } X \geqslant 0 \\ -X, & \text{when } X < 0. \end{cases}$
So

$$\begin{aligned}
\mathrm{P}(W = w) &= p(w) + p(-w), \qquad w = 1, 2, ... \\
\text{and} \quad \mathrm{P}(W = 0) &= p(0).
\end{aligned}$$

(c) $Z = \text{sgn}(X) = \begin{cases} 1, & \text{when } X > 0 \\ -1, & \text{when } X < 0 \\ 0, & \text{when } X = 0. \end{cases}$
So

$$\begin{aligned}
\mathrm{P}(Z = 1) &= \sum_{x>0} p(x), \\
\mathrm{P}(Z = 0) &= p(0), \\
\mathrm{P}(Z = -1) &= \sum_{x<0} p(x).
\end{aligned}$$

/continued overleaf

(ii) Let

$$\begin{aligned} g(a) &= \mathrm{E}[(X-a)^2] \\ &= \mathrm{E}[X^2 - 2aX + a^2] \\ &= \mathrm{E}(X^2) - 2a\mathrm{E}(X) + a^2. \end{aligned}$$

Then

$$\begin{aligned} \frac{dg}{da} &= -2\mathrm{E}(X) + 2a = 0 \qquad \text{when } a = \mathrm{E}(X) \\ \frac{d^2g}{da^2} &= 2 \;\; > 0 \qquad \text{when } a = \mathrm{E}(X) \end{aligned}$$

So $\mathrm{E}[(X-a)]^2$ is *min*imized when $a = \mathrm{E}(X)$.
[Note: $\mathrm{E}(|X-b|)$ is minimized when $b =$median of X.]

3. (i) (a) We have

$$\begin{aligned} \mathrm{P}(X > m) &= \sum_{x=m+1}^{\infty} \mathrm{P}(X = x) \\ &= pq^m + pq^{m+1} + \cdots \\ &= pq^m[1 + q + q^2 + \cdots] \\ &= \frac{pq^m}{1-q} = q^m \quad \text{since } p + q = 1. \end{aligned}$$

(b) 'No memory' property:

$$\begin{aligned} \mathrm{P}(X > m+n | X > m) &= \frac{\mathrm{P}((X > m+n) \cap (X > m))}{\mathrm{P}(X > m)} \\ &= \frac{\mathrm{P}(X > m+n)}{\mathrm{P}(X > m)} \quad \text{since } (X > m+n) \subset (X > m) \\ &= \frac{q^{m+n}}{q^m} = q^n = \mathrm{P}(X > n). \quad \text{[using result in part (a)]} \end{aligned}$$

(ii) We can write out $\mathrm{E}(X)$ as follows:

$$\begin{aligned} \mathrm{E}(X) &= \sum_{x=0}^{\infty} x\mathrm{P}(X = x) \\ &= \sum_{x=1}^{\infty} x\mathrm{P}(X = x) \\ &= \quad \mathrm{P}(X=1) \\ &\quad + \mathrm{P}(X=2) \quad + \mathrm{P}(X=2) \\ &\quad + \mathrm{P}(X=3) \quad + \mathrm{P}(X=3) \quad + \mathrm{P}(X=3) \\ &\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{aligned}$$

Since the series is convergent and the terms are positive, we can re-arrange the order of the terms. So, summing vertically, we obtain

$$\mathrm{E}(X) = \sum_{x=0}^{\infty} \mathrm{P}(X > x).$$

For the geometric distribution in part (i):

$$\begin{aligned} \mathrm{P}(X > 0) &= 1; \\ \mathrm{P}(X > x) &= q^x \quad (x \text{ a positive integer}). \end{aligned}$$

So

$$\mathrm{E}(X) = 1 + q + q^2 + \cdots = \frac{1}{1-q} = \frac{1}{p}.$$

4. We are given that X and Y are independent random variables with

$$\mathrm{P}(X=k)=\mathrm{P}(Y=k)=pq^k, \quad k=0,1,...; \ \ p+q=1.$$

Then

(a)
$$\begin{aligned}
\mathrm{P}(X=Y) &= \sum_{k=0}^{\infty}\mathrm{P}(X=k,Y=k)\\
&= \sum_{k=0}^{\infty}\mathrm{P}(X=k).\mathrm{P}(Y=k) \qquad \text{[since } X \text{ and } Y \text{ are independent]}\\
&= \sum_{k=0}^{\infty}\left(pq^k\right)^2\\
&= p^2\sum_{k=0}^{\infty}(q^2)^k\\
&= \frac{p^2}{(1-q^2)} \qquad \text{[since } 0<q^2<1\text{]}\\
&= \frac{p}{(1+q)}. \qquad \text{[since } p=1-q\text{]}
\end{aligned}$$

(b) We have that

$$\begin{aligned}
\mathrm{P}(X+Y=n) &= \sum_{x=0}^{n}\mathrm{P}(X=x,Y=n-x)\\
&= \sum_{x=0}^{n}\mathrm{P}(X=x).\mathrm{P}(Y=n-x) \qquad \text{[independence]}\\
&= \sum_{x=0}^{n}pq^x.pq^{n-x}\\
&= \sum_{x=0}^{n}p^2q^n=(n+1)p^2q^n. \qquad (*)
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{P}(X=x|X+Y=n) &= \frac{\mathrm{P}(X=x \text{ and } X+Y=n)}{\mathrm{P}(X+Y=n)}\\
&= \frac{\mathrm{P}(X=x,Y=n-x)}{\mathrm{P}(X+Y=n)}\\
&= \frac{pq^x.pq^{n-x}}{(n+1)p^2q^n} \qquad \text{[using independence and (*)]}\\
&= \frac{1}{n+1}, \quad x=0,1,...,n
\end{aligned}$$

– the discrete uniform distribution on (0,1,2,...,n)

(c) $U=\min(X,Y)$ takes the values 0,1,2,...
The event $(U=u)$ can be decomposed into mutually exclusive events thus:

$$\begin{aligned}
(U=u) &= (X=u,Y=u)\\
&\quad \cup(X=u,Y=u+1)\cup(X=u,Y=u+2)\cup\cdots\\
&\quad \cup(X=u+1,Y=u)\cup(X=u+2,Y=u)\cup\cdots
\end{aligned}$$

/continued overleaf

So, invoking the most general ('countably additive') form of Axiom 3, we have:

$$\begin{aligned}
\mathrm{P}(U=u) &= \mathrm{P}(U=u, Y=u) + \sum_{y=u+1}^{\infty} \mathrm{P}(X=u, Y=y) + \sum_{x=u+1}^{\infty} \mathrm{P}(Y=u, X=x) \\
&= (pq^u)^2 + \sum_{y=u+1}^{\infty} pq^u.pq^y + \sum_{x=u+1}^{\infty} pq^u.pq^x \\
&= p^2q^{2u} + 2\sum_{y=u+1}^{\infty} p^2q^{u+y} \\
&= p^2q^{2u} + 2p^2q^{2u+1}\sum_{i=0}^{\infty} q^i \\
&= p^2q^{2u} + 2p^2q^{2u+1}/(1-q) = p^2q^{2u} + 2pq^{2u+1} \\
&= pq^{2u}(p+2q) = pq^{2u}(q+1), \qquad u = 0, 1, ...
\end{aligned}$$

5. (a)

Sample Points					X	Y	Z
H	H	H	H	H	5	1	5
H	H	H	H	T	4	1	4
H	H	H	T	H	4	2	3
H	H	T	H	H	4	2	2
H	T	H	H	H	4	2	3
T	H	H	H	H	4	1	4
H	H	H	T	T	3	1	3
H	H	T	H	T	3	2	2
H	T	H	H	T	3	2	2
T	H	H	H	T	3	1	3
H	H	T	T	H	3	2	2
H	T	H	T	H	3	3	1
T	H	H	T	H	3	2	2
H	T	T	H	H	3	2	2
T	H	T	H	H	3	2	2
T	T	H	H	H	3	1	3
H	H	T	T	T	2	1	2
H	T	H	T	T	2	2	1
T	H	H	T	T	2	1	2
H	T	T	H	T	2	2	1
T	H	T	H	T	2	2	1
T	T	H	H	T	2	1	2
H	T	T	T	H	2	2	1
T	H	T	T	H	2	2	1
T	T	H	T	H	2	2	1
T	T	T	H	H	2	1	2
H	T	T	T	T	1	1	1
T	H	T	T	T	1	1	1
T	T	H	T	T	1	1	1
T	T	T	H	T	1	1	1
T	T	T	T	H	1	1	1
T	T	T	T	T	0	0	0

/continued overleaf

(b) Let

$$p(x,y,z) = \mathrm{P}(X=x, Y=y, Z=z).$$

From the listing of the sample space in (a), we deduce that

$$\begin{array}{ll} p(0,0,0) = \frac{1}{32} & p(1,1,1) = \frac{5}{32} \\ p(2,1,2) = \frac{4}{32} & p(2,2,1) = \frac{6}{32} \\ p(3,1,3) = \frac{3}{32} & p(3,2,2) = \frac{6}{32} \\ p(3,3,1) = \frac{1}{32} & p(4,1,4) = \frac{2}{32} \\ p(4,2,2) = \frac{1}{32} & p(4,2,3) = \frac{2}{32} \\ p(5,1,5) = \frac{1}{32} & \end{array}$$

All other probabilities of the form

$$p(x,y,z),\ 0 \leqslant x \leqslant 5,\ 0 \leqslant y \leqslant 3,\ 0 \leqslant z \leqslant 5$$

are zero. [Check: $\sum_{x,y,z} p(x,y,z) = 1$.]

The joint probability function of (X, Y) is given by

$$\begin{aligned} \mathrm{P}(X=x, Y=y) &= \sum_{z=0}^{5} \mathrm{P}(X=x, Y=y, Z=z) \\ &= \sum_{z=0}^{5} p(x,y,z) \qquad \text{for} \quad 0 \leqslant x \leqslant 5, \quad 0 \leqslant y \leqslant 3. \end{aligned}$$

If this function is tabulated in a two-way table, the row and column totals give the probability function values for the random variables Y and X respectively, since

$$\begin{aligned} \mathrm{P}(X=x) &= \sum_{y=0}^{3} \mathrm{P}(X=x, Y=y) = \sum_{y=0}^{3}\sum_{z=0}^{5} p(x,y,z), \quad 0 \leqslant x \leqslant 5 \\ \text{and} \quad \mathrm{P}(Y=y) &= \sum_{x=0}^{5} \mathrm{P}(X=x, Y=y) = \sum_{x=0}^{5}\sum_{z=0}^{5} p(x,y,z), \quad 0 \leqslant y \leqslant 3. \end{aligned}$$

Thus:

		X						
		0	1	2	3	4	5	
	0	$\frac{1}{32}$	0	0	0	0	0	$\frac{1}{32}$
	1	0	$\frac{5}{32}$	$\frac{4}{32}$	$\frac{3}{32}$	$\frac{2}{32}$	$\frac{1}{32}$	$\frac{15}{32}$
Y	2	0	0	$\frac{6}{32}$	$\frac{6}{32}$	$\frac{3}{32}$	0	$\frac{15}{32}$
	3	0	0	0	$\frac{1}{32}$	0	0	$\frac{1}{32}$
		$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$	1

Similarly, the joint probability function of (Y, Z) and the probability functions of Y and Z are

		Z						
		0	1	2	3	4	5	
	0	$\frac{1}{32}$	0	0	0	0	0	$\frac{1}{32}$
	1	0	$\frac{5}{32}$	$\frac{4}{32}$	$\frac{3}{32}$	$\frac{2}{32}$	$\frac{1}{32}$	$\frac{15}{32}$
Y	2	0	$\frac{6}{32}$	$\frac{7}{32}$	$\frac{2}{32}$	0	0	$\frac{15}{32}$
	3	0	$\frac{1}{32}$	0	0	0	0	$\frac{1}{32}$
		$\frac{1}{32}$	$\frac{12}{32}$	$\frac{11}{32}$	$\frac{5}{32}$	$\frac{2}{32}$	$\frac{1}{32}$	1

/continued overleaf

(c) $\mathrm{E}(Y) = (0 \times \frac{1}{32}) + (1 \times \frac{15}{32}) + (2 \times \frac{15}{32}) + (3 \times \frac{1}{32}) = \frac{48}{32} = \frac{3}{2}$

$\mathrm{E}(Y^2) = (0 \times \frac{1}{32}) + (1 \times \frac{15}{32}) + (4 \times \frac{15}{32}) + (9 \times \frac{1}{32}) = \frac{84}{32} = \frac{21}{8}$

So $\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \{\mathrm{E}(Y)\}^2 = \frac{21}{8} - \frac{9}{4} = \frac{3}{8}$.

$\mathrm{E}(Z) = (0 \times \frac{1}{32}) + (1 \times \frac{12}{32}) + (2 \times \frac{11}{32}) + (3 \times \frac{5}{32}) + (4 \times \frac{2}{32}) + (5 \times \frac{1}{32}) = \frac{62}{32} = \frac{31}{16}$

$\mathrm{E}(Z^2) = (0 \times \frac{1}{32}) + (1 \times \frac{12}{32}) + (4 \times \frac{11}{32}) + (9 \times \frac{5}{32}) + (16 \times \frac{2}{32}) + (25 \times \frac{1}{32}) = \frac{158}{32} = \frac{79}{16}$

So $\mathrm{Var}(Z) = \mathrm{E}(Z^2) - \{\mathrm{E}(Z)\}^2 = \frac{79}{16} - \left(\frac{31}{16}\right)^2 = \frac{303}{256}$.

$$\begin{aligned}\mathrm{E}(YZ) &= (1 \times 1 \times \tfrac{5}{32}) + (1 \times 2 \times \tfrac{4}{32}) + (1 \times 3 \times \tfrac{3}{32}) + (1 \times 4 \times \tfrac{2}{32}) + (1 \times 5 \times \tfrac{1}{32}) \\ &\quad + (2 \times 1 \times \tfrac{6}{32}) + (2 \times 2 \times \tfrac{7}{32}) + (2 \times 3 \times \tfrac{2}{32}) + (3 \times 1 \times \tfrac{1}{32}) \\ &= \tfrac{90}{32}.\end{aligned}$$

Hence

$$\mathrm{Cov}(Y, Z) = \tfrac{90}{32} - \tfrac{3}{2} \times \tfrac{31}{16} = -\tfrac{3}{32}$$

and

$$\rho(Y, Z) = \frac{-\frac{3}{32}}{\sqrt{\frac{3}{8} \times \frac{303}{256}}} = -\frac{3\sqrt{2}}{\sqrt{909}} \approx \mathbf{-0.141}$$

(d) $\mathrm{P}(X = x | Y = 1) = \mathrm{P}(X = x, Y = 1)/\mathrm{P}(Y = 1), \quad 0 \leqslant x \leqslant 5.$

x	0	1	2	3	4	5	
$\mathrm{P}(X = x \mid Y = 1)$	0	$\frac{5}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	1

$$\mathrm{P}(Y = y, Z = z | X = 3) = \frac{\mathrm{P}(X = 3, Y = y, Z = z)}{\mathrm{P}(X = 3)}, \quad 0 \leqslant y \leqslant 3, 0 \leqslant z \leqslant 5.$$

Thus

$$\begin{aligned}\mathrm{P}(Y = 1, Z = 3 | X = 3) &= \tfrac{3}{10} \\ \mathrm{P}(Y = 2, Z = 2 | X = 3) &= \tfrac{6}{10} \\ \mathrm{P}(Y = 3, Z = 1 | X = 3) &= \tfrac{1}{10};\end{aligned}$$

all other probabilities are zero.

(e) The random variable $\mathrm{E}(Y|X)$ takes the values $\mathrm{E}(Y|X = x), \quad x = 0, ..., 5$, where

$$\begin{aligned}\mathrm{E}(Y|X = x) &= \sum_{y=0}^{3} y\mathrm{P}(Y = y|X = x) \\ &= \sum_{y=0}^{3} y\frac{\mathrm{P}(X = x, Y = y)}{\mathrm{P}(X = x)}.\end{aligned}$$

Thus:

$$\begin{aligned}\mathrm{E}(Y|X = 0) &= 0 \times \tfrac{1/32}{1/32} = 0 \\ \mathrm{E}(Y|X = 1) &= 1 \times \tfrac{5/32}{5/32} = 1 \\ \mathrm{E}(Y|X = 2) &= 1 \times \tfrac{4/32}{10/32} + 2 \times \tfrac{6/32}{10/32} = \tfrac{16}{10} \\ \mathrm{E}(Y|X = 3) &= 1 \times \tfrac{3/32}{10/32} + 2 \times \tfrac{6/32}{10/32} + 3 \times \tfrac{1/32}{10/32} = \tfrac{18}{10} \\ \mathrm{E}(Y|X = 4) &= 1 \times \tfrac{2/32}{5/32} + 2 \times \tfrac{3/32}{5/32} = \tfrac{16}{10} \\ \mathrm{E}(Y|X = 5) &= 1 \times \tfrac{1/32}{1/32} = 1.\end{aligned}$$

The probabilities are derived from $\{\mathrm{P}(X = x)\}$. Thus:

$\mathrm{E}(Y\mid X)$	0	1	$\frac{16}{10}$	$\frac{18}{10}$	
Prob.	$\frac{1}{32}$	$\frac{6}{32}$	$\frac{10}{32} + \frac{5}{32} = \frac{15}{32}$	$\frac{10}{32}$	1

and $\mathrm{E}[\mathrm{E}(Y|X)] = (0 \times \frac{1}{32}) + (1 \times \frac{6}{32}) + (\frac{16}{10} \times \frac{15}{32}) + (\frac{18}{10} \times \frac{10}{32}) = \frac{48}{32} = \mathrm{E}(Y)$.

6. Let

$X =$ time to freedom (hours)
$Y =$ number of door originally chosen (1,2 or 3).

Then

$$\mathrm{E}(X) = \mathrm{E}(X|Y=1)\mathrm{P}(Y=1) + \mathrm{E}(X|Y=2)\mathrm{P}(Y=2) + \mathrm{E}(X|Y=3)\mathrm{P}(Y=3).$$

Now

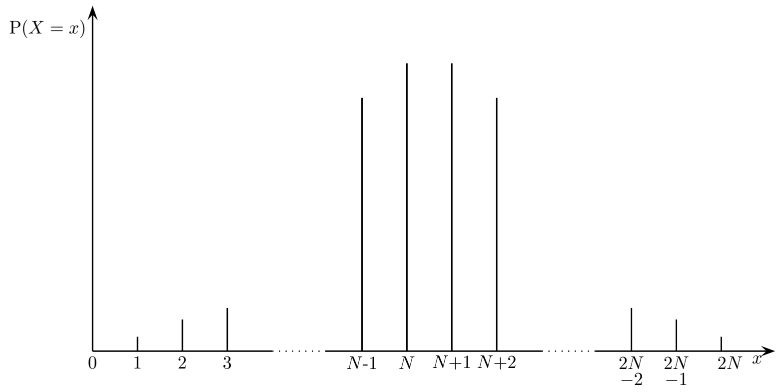
$$\begin{aligned} \mathrm{E}(X|Y=1) &= 2 + \mathrm{E}(X) \\ \mathrm{E}(Y|Y=2) &= 4 + \mathrm{E}(X) \\ \mathrm{E}(X|Y=3) &= 1. \end{aligned}$$

So

$$\begin{aligned} \mathrm{E}(X) &= 0.5[2+\mathrm{E}(X)] + 0.3[4+\mathrm{E}(X)] + 0.2[1] \\ &= 2.4 + 0.8\mathrm{E}(X). \end{aligned}$$

So $\mathrm{E}(X) = 12.0$ days.

7. (a)



$\mathrm{P}(X=x) = \mathrm{P}(X = 2N - x + 1), \qquad x = 1, ..., N$

$$\mathrm{E}(X - N - \tfrac{1}{2}) = \sum_{x=1}^{2N} (x - N - \tfrac{1}{2})\mathrm{P}(X=x).$$

Now

$$\begin{aligned} \text{1st term} &= (\tfrac{1}{2} - N)\mathrm{P}(X=1) \\ \text{2nd term} &= (\tfrac{3}{2} - N)\mathrm{P}(X=2) \\ &\quad \text{etc.} \end{aligned}$$

$$\begin{aligned} \text{2nd last term} &= (N - \tfrac{3}{2})\mathrm{P}(X = 2N-1) &= (N - \tfrac{3}{2})\mathrm{P}(X=2) \\ \text{last term} &= (N - \tfrac{1}{2})\mathrm{P}(X = 2N) &= (N - \tfrac{1}{2})\mathrm{P}(X=1). \end{aligned}$$

Summing 1st and last terms gives 0;
2nd and 2nd last terms gives 0;
etc.

So

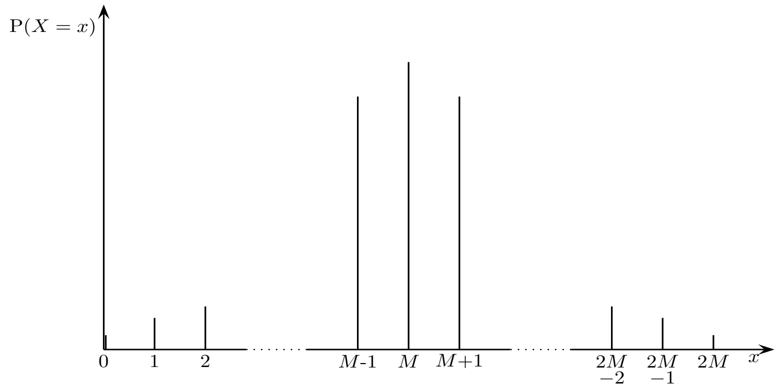
$$\mathrm{E}(X - N - \tfrac{1}{2}) = 0 = \mathrm{E}(X) - (N + \tfrac{1}{2}),$$

i.e. $$\mathrm{E}(X) = N + \tfrac{1}{2}.$$

Since $\mathrm{P}(X \leqslant N + \frac{1}{2}) = \mathrm{P}(X \geqslant N + \frac{1}{2}) = \frac{1}{2}$, the median of X is $N + \frac{1}{2}$ (actually *any* value between N and $N+1$ can be considered the median).

/continued overleaf

(b)



$$\mathrm{P}(Y=y)=\mathrm{P}(Y=2M-y), \qquad y=0,...,M.$$

By a similar procedure to that in part (a), we can show that

$$\mathrm{E}(Y-M)=0$$

and hence

$$\mathrm{E}(Y)=M.$$

Since $\mathrm{P}(Y<M)=P(Y>M)$, the median of Y is M.

8. We have:

$$\begin{aligned}\mathrm{P}(X_1=X_2) &= \mathrm{P}\left(\bigcup_{k=0}^{n}[X_1=k,X_2=k]\right)\\ &= \sum_{k=0}^{n}\mathrm{P}(X_1=k,X_2=k) && \text{[m.e. events]}\\ &= \sum_{k=0}^{n}\mathrm{P}(X_1=k)\mathrm{P}(X_2=k). && \text{[independence]}\end{aligned}$$

But

$$\mathrm{P}(X_2=k)=\mathrm{P}(X_2=n-k) \qquad \text{since} \quad \mathrm{P}(H)=\mathrm{P}(T).$$

So

$$\begin{aligned}\mathrm{P}(X_1=X_2) &= \sum_{k=0}^{n}\mathrm{P}(X_1=k)\mathrm{P}(X_2=n-k)\\ &= \sum_{k=0}^{n}\mathrm{P}(X_1=k,X_2=n-k) && \text{[independence]}\\ &= \mathrm{P}\left(\bigcup_{k=0}^{n}[X_1=k,X_2=n-k]\right) && \text{[m.e. events]}\\ &= \mathrm{P}(X_1+X_2=n).\end{aligned}$$

Because the two people toss independently, the experiment can be regarded as a single Bernoulli process with $2n$ trials and $p=\frac{1}{2}$, $X=X_1+X_2$ being the total number of heads obtained. Then the required probability is

$$\begin{aligned}\mathrm{P}(X=n) &= \binom{2n}{n}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{2n-n} && \text{[binomial distribution]}\\ &= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}.\end{aligned}$$

9. (a)

		X=0	1	2	3	4	5	
	0	$\frac{1}{32}$	0	0	0	0	0	$\frac{1}{32}$
	1	0	$\frac{5}{32}$	$\frac{6}{32}$	$\frac{1}{32}$	0	0	$\frac{12}{32}$
Z	2	0	0	$\frac{4}{32}$	$\frac{6}{32}$	$\frac{1}{32}$	0	$\frac{11}{32}$
	3	0	0	0	$\frac{3}{32}$	$\frac{2}{32}$	0	$\frac{5}{32}$
	4	0	0	0	0	$\frac{2}{32}$	0	$\frac{2}{32}$
	5	0	0	0	0	0	$\frac{1}{32}$	$\frac{1}{32}$
		$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$	1

(b) $\mathrm{E}(X) = (0 \times \frac{1}{32}) + (1 \times \frac{5}{32}) + (2 \times \frac{10}{32}) + (3 \times \frac{10}{32}) + (4 \times \frac{5}{32}) + (5 \times \frac{1}{32}) = \frac{80}{32} = \frac{5}{2}$
$\mathrm{E}(X^2) = (0\times\frac{1}{32})+(1\times\frac{5}{32})+(4\times\frac{10}{32})+(9\times\frac{10}{32})+(16\times\frac{5}{32})+(25\times\frac{1}{32}) = \frac{240}{32} = \frac{15}{2}.$
So $\mathrm{Var}(X) = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2 = \frac{15}{2} - \left(\frac{5}{2}\right)^2 = \frac{5}{4}.$

$$\begin{aligned}\mathrm{E}(XY) &= (1 \times 1 \times \tfrac{5}{32}) + (2 \times 1 \times \tfrac{4}{32}) + (2 \times 2 \times \tfrac{6}{32}) + (3 \times 1 \times \tfrac{3}{32}) + (3 \times 2 \times \tfrac{6}{32}) \\ &\quad +(3 \times 3 \times \tfrac{1}{32}) + (4 \times 1 \times \tfrac{2}{32}) + (4 \times 2 \times \tfrac{3}{32}) + (5 \times 1 \times \tfrac{1}{32}) \\ &= \tfrac{128}{32} = 4\end{aligned}$$

$$\begin{aligned}\mathrm{E}(XZ) &= (1 \times 1 \times \tfrac{5}{32}) + (2 \times 1 \times \tfrac{6}{32}) + (2 \times 2 \times \tfrac{4}{32}) + (3 \times 1 \times \tfrac{1}{32}) + (3 \times 3 \times \tfrac{3}{32}) \\ &\quad +(4 \times 2 \times \tfrac{1}{32}) + (4 \times 3 \times \tfrac{2}{32}) + (4 \times 4 \times \tfrac{2}{32}) + (5 \times 5 \times \tfrac{1}{32}) \\ &= \tfrac{188}{32} = \tfrac{47}{8}.\end{aligned}$$

So

$$\begin{aligned}\mathrm{Cov}(X,Y) &= 4 - \tfrac{5}{2} \times \tfrac{3}{2} = \tfrac{1}{4}, \\ \rho(X,Y) &= \frac{\frac{1}{4}}{\sqrt{\frac{5}{4} \times \frac{3}{8}}} = \sqrt{\tfrac{2}{15}} \approx \mathbf{0.365}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(X,Z) &= \tfrac{188}{32} - \tfrac{5}{2} \times \tfrac{31}{16} = \tfrac{33}{32}, \\ \rho(X,Z) &= \frac{\frac{33}{32}}{\sqrt{\frac{5}{4} \times \frac{303}{256}}} = \frac{33}{\sqrt{1515}} \approx \mathbf{0.848}\end{aligned}$$

(c) $\mathrm{P}(X = x|Y = 2, Z = 2) = \dfrac{\mathrm{P}(X = x, Y = 2, Z = 2)}{\mathrm{P}(Y = 2, Z = 2)}, \quad 0 \leqslant x \leqslant 5.$

So

$$\begin{aligned}\mathrm{P}(X = 3|Y = 2, Z = 2) &= \tfrac{6}{7} \\ \mathrm{P}(X = 4|Y = 2, Z = 2) &= \tfrac{1}{7};\end{aligned}$$

all other probabilities are zero.

(d)

$$\mathrm{E}(Z|X = x) = \sum_{z=0}^{5} z\frac{\mathrm{P}(X = x, Z = z)}{\mathrm{P}(X = x)}.$$

Thus:

$$\begin{aligned}\mathrm{E}(Z|X = 0) &= 0 \times \tfrac{1/32}{1/32} = 0 \\ \mathrm{E}(Z|X = 1) &= 1 \times \tfrac{5/32}{5/32} = 1 \\ \mathrm{E}(Z|X = 2) &= 1 \times \tfrac{6/32}{10/32} + 2 \times \tfrac{4/32}{10/32} = \tfrac{14}{10} \\ \mathrm{E}(Z|X = 3) &= 1 \times \tfrac{1/32}{10/32} + 2 \times \tfrac{6/32}{10/32} + 3 \times \tfrac{3/32}{10/32} = \tfrac{22}{10} \\ \mathrm{E}(Z|X = 4) &= 2 \times \tfrac{1/32}{5/32} + 3 \times \tfrac{2/32}{5/32} + 4 \times \tfrac{2/32}{5/32} = \tfrac{32}{10} \\ \mathrm{E}(Z|X = 5) &= 5 \times \tfrac{1/32}{1/32} = 5.\end{aligned}$$

/continued overleaf

From $\{\mathrm{P}(X=x)\}$ we deduce

$\mathrm{E}(Z\|X)$	0	1	$\frac{14}{10}$	$\frac{22}{10}$	$\frac{32}{10}$	5	
Prob.	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$	1

So

$$\begin{aligned}\mathrm{E}[\mathrm{E}(Z|X)] &= (0\times\tfrac{1}{32})+(1\times\tfrac{5}{32})+(\tfrac{14}{10}\times\tfrac{10}{32})+(\tfrac{22}{10}\times\tfrac{10}{32})+(\tfrac{32}{10}\times\tfrac{5}{32})+(5\times\tfrac{1}{32})\\ &= \tfrac{62}{32}=\mathrm{E}(Z).\end{aligned}$$

10. Let

$X_k =$ return when critical value k is used
$S =$ value on first roll.

The probability distribution of S is:

j	2	3	4	5	6	7	8	9	10	11	12	
$\mathrm{P}(S=j)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

Then

$$\mathrm{E}(X_k)=\sum_{j=2}^{12}\mathrm{E}(X_k|S=j)\mathrm{P}(S=j).$$

Now

$$\mathrm{E}(X_k|S=j)=\begin{cases}0, & \text{if } j=7\\ \mathrm{E}(X_k), & \text{if } j<k \text{ and } j\neq 7 \text{ (because we start again)}\\ j, & \text{if } j\geqslant k \text{ and } j\neq 7.\end{cases}$$

Then

$$\begin{aligned}\mathrm{E}(X_6) &= \tfrac{1}{36}[1+2+3+4]\mathrm{E}(X_6)+\tfrac{1}{36}[(6\times5)+(8\times5)+(9\times4)\\ &\qquad +(10\times3)+(11\times2)+(12\times1)]\\ &= \tfrac{10}{36}\mathrm{E}(X_6)+\tfrac{170}{36}\end{aligned}$$

i.e $\quad \frac{26}{36}\mathrm{E}(X_6)=\frac{170}{36} \quad\longrightarrow\quad \mathrm{E}(X_6)=\frac{170}{36}=\mathbf{6.538}.$

$$\begin{aligned}[\mathrm{E}(X_7)=]\mathrm{E}(X_8) &= \tfrac{1}{36}[1+2+3+4+5]\mathrm{E}(X_8)+\tfrac{1}{36}[(8\times5)+(9\times4)\\ &\qquad +(10\times3)+(11\times2)+(12\times1)]\\ &= \tfrac{15}{36}\mathrm{E}(X_8)+\tfrac{140}{36}\end{aligned}$$

i.e $\quad \frac{21}{36}\mathrm{E}(X_8)=\frac{140}{36} \quad\longrightarrow\quad \mathrm{E}(X_8)=\frac{140}{21}=\mathbf{6.667}.$

$$\begin{aligned}\mathrm{E}(X_9) &= \tfrac{1}{36}[1+2+3+4+5+5]\mathrm{E}(X_9)+\tfrac{1}{36}[(9\times4)+(10\times3)+(11\times2)+(12\times1)]\\ &= \tfrac{20}{36}\mathrm{E}(X_9)+\tfrac{100}{36}\end{aligned}$$

i.e $\quad \frac{16}{36}\mathrm{E}(X_9)=\frac{100}{36} \quad\longrightarrow\quad \mathrm{E}(X_9)=\frac{100}{16}=\mathbf{6.250}.$

So $\mathrm{E}(X_k)$ is maximised at $k=8$.