SOR201 Solutions to Examples 4

1. (i) (a) $I_i = \begin{cases} 1, & \text{if a success occurs on the } i^{\text{th}} \text{ trial} \\ 0, & \text{otherwise.} \end{cases}$

So

$$P(I_i = 1) = p, \qquad P(I_i = 0) = 1 - p = q,$$
$$E(I_i) = 0 \times q + 1 \times p = p,$$
$$E(I_i^2) = 0^2 \times q + 1^2 \times p = p,$$

and $\text{Var}(I_i) = p - p^2 = pq$.

$I_1, ..., I_n$ are independent random variables because I_i is associated with the outcome of the i^{th} trial and the trials are independent, i.e. their outcomes are independent events.

(b) $X = I_1 + \cdots + I_n =$ number of successes in the n trials.

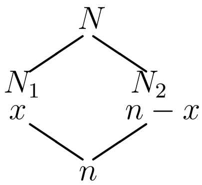
$X \sim \text{Binomial}(n, p)$.

$$\begin{aligned} E(X) &= E(I_1 + \cdots + I_n) &&= E(I_1) + \cdots E(I_n) &&= np, \\ \text{Var}(X) &= \text{Var}(I_1 + \cdots + I_n) &&= \text{Var}(I_1) + \cdots + \text{Var}(I_n) &&= npq. \end{aligned}$$

(There are no covariance terms involved because $I_1, ..., I_n$ are independent random variables.)

(ii) (a) The random variable X has the hypergeometric distribution

$$P(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}},$$



where $\quad 0 \leqslant x \leqslant N_1, \quad 0 \leqslant n - x \leqslant N_2.$

$X = I_1 + \cdots + I_n$.

(b) Since I_i and I_j are indicator random variables,

$$\begin{aligned} E(I_i) &= P(I_i = 1) = \frac{N_1}{N}, \text{ [because the } i^{\text{th}} \text{ ball selected is equally likely to be any of the } N \text{ balls]} \\ \text{Var}(I_i) &= E(I_i^2) - [E(I_i)]^2 \\ &= P(I_i = 1) - \left(\frac{N_1}{N}\right)^2 = \frac{N_1}{N} - \left(\frac{N_1}{N}\right)^2, \\ E(I_i.I_j) &= 0 \times 0 \times P(I_i = 0, I_j = 0) \\ &\quad + 0 \times 1 \times P(I_i = 0, I_j = 1) \\ &\quad + 1 \times 0 \times P(I_i = 1, I_j = 0) \\ &\quad + 1 \times 1 \times P(I_i = 1, I_j = 1) \\ &= P(I_i = 1, I_j = 1) \\ &= P(I_j = 1 | I_i = 1) P(I_i = 1) = \frac{N_1 - 1}{N - 1} \times \frac{N_1}{N}, \end{aligned}$$

so

$$\begin{aligned} \text{Cov}(I_i, I_j) &= E(I_i.I_j) - E(I_i)E(I_j) \\ &= \frac{N_1(N_1 - 1)}{N(N - 1)} - \left(\frac{N_1}{N}\right)^2 \\ &= -\frac{N_1 N_2}{N^2(N - 1)}. \qquad [N_1 + N_2 = N] \end{aligned}$$

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Then $\quad \mathrm{E}(X) = \mathrm{E}(I_1 + \cdots + I_n) = \mathrm{E}(I_1) + \cdots + \mathrm{E}(I_n) = \dfrac{nN_1}{N},$

and

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{Var}(I_1 + \cdots + I_n) = \sum_{i=1}^{n} \mathrm{Var}(I_i) + 2\sum_{i<j} \mathrm{Cov}(I_i, I_j) \\
&= n\left\{\frac{N_1}{N} - \left(\frac{N_1}{N}\right)^2\right\} + 2\binom{n}{2}\left\{-\frac{N_1 N_2}{N^2(N-1)}\right\} \\
&= \frac{nN_1}{N}\left\{1 - \frac{N_1}{N} - \frac{(n-1)N_2}{N(N-1)}\right\} \\
&= \frac{nN_1}{N^2}\left\{N_2 - \frac{(n-1)N_2}{(N-1)}\right\} = \frac{nN_1N_2(N-n)}{N^2(N-1)}.
\end{aligned}$$

(iii) Let

$$I_i = \begin{cases} 1, & \text{if } A_i \text{ occurs} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\quad \mathrm{E}(I_i) = \mathrm{P}(I_i = 1) = \mathrm{P}(A_i) = \dfrac{(n-1)!}{n!} = \dfrac{1}{n}, \quad i = 1, \ldots, n$

and $\quad \mathrm{Var}(I_i) = \mathrm{P}(I_i = 1) - [\mathrm{P}(I_i = 1)]^2 = \dfrac{1}{n} - \left(\dfrac{1}{n}\right)^2, \quad i = 1, \ldots, n.$

Now $\quad S_n = I_1 + I_2 + \cdots + I_n.$

So $\quad \mathrm{E}(S_n) = \displaystyle\sum_{i=1}^{n} \mathrm{E}(I_i) = n \times \frac{1}{n} = 1.$

Also $\quad \mathrm{Var}(S_n) = \displaystyle\sum_{i=1}^{n} \mathrm{Var}(I_i) + 2\sum_{i<j} \mathrm{Cov}(I_i, I_j),$

where

$$\begin{aligned}
\mathrm{Cov}(I_i, I_j) &= \mathrm{E}(I_i.I_j) - \mathrm{E}(I_i)\mathrm{E}(I_j) \\
&= \mathrm{P}(A_i \cap A_j) - \mathrm{P}(A_i)\mathrm{P}(A_j) \qquad i \neq j \\
&= \frac{(n-2)!}{n!} - \left(\frac{1}{n}\right)^2 = \frac{1}{n(n-1)} - \frac{1}{n^2}.
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{Var}(S_n) &= n\left(\frac{1}{n} - \frac{1}{n^2}\right) + 2\binom{n}{2}\left(\frac{1}{n(n-1)} - \frac{1}{n^2}\right) \\
&= 1 - \frac{1}{n} + 1 - \frac{n(n-1)}{n^2} = 1.
\end{aligned}$$

Note:

(a) In deriving expressions for $\mathrm{E}(I_i)$ and $\mathrm{Cov}(I_i, I_j)$, one can alternatively argue as in part (ii)(b).

(b) From Examples 1, Question 8, we have that

$$\mathrm{P}(S_n = k) \longrightarrow \frac{e^{-1}}{k!} \quad \text{as } n \to \infty$$

i.e. S_n has a Poisson distribution with

$$\mathrm{E}(S_n) = \mathrm{Var}(S_n) = 1.$$

Now we have found that this result holds for *all* n (even though the *distribution* of S_n for finite n is not Poisson).

2. (i) The PGF of X is

$$\begin{aligned} G_X(s) &= \mathrm{E}(s^X) = \sum_x \mathrm{P}(X=x)s^x \\ &= \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} s^x \\ &= \sum_{x=0}^{n} \binom{n}{x} (ps)^x q^{n-x} = (ps+q)^n. \end{aligned}$$

Then

$$G_X^{(1)}(s) = n(ps+q)^{n-1}.p \quad \text{so} \quad \mathrm{E}(X) = G_X^{(1)}(1) = np.$$

$$G_X^{(2)}(s) = n(n-1)(ps+q)^{n-2}.p^2 \quad \text{so} \quad G_X^{(2)}(1) = n(n-1)p^2.$$

So

$$\begin{aligned} \mathrm{Var}(X) &= G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1)\right]^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= -np^2 + np = np(1-p) = npq. \end{aligned}$$

Now

$$\begin{aligned} G_{X+Y}(s) &= G_X(s).G_Y(s) \quad \text{when X,Y are independent} \\ &= (ps+q)^n.(ps+q)^m \\ &= (ps+q)^{m+n} \quad - \text{ the PGF of Bin}(m+n,p). \end{aligned}$$

So $\quad X+Y \sim \mathrm{Bin}(m+n,p).$

Alternative argument:

Consider n independent Bernoulli trials, each with probability p, followed by m independent Bernoulli trials, also each with probability p.

Then

number of successes in first n trials
+ number of successes in last m trials = number of successes in $(n+m)$ trials.

Hence $\quad X+Y = Z \sim \mathrm{Bin}(n+m,p).$

(ii) $$G_X(s) = \frac{1-s^{M+1}}{(M+1)(1-s)}.$$

Since $\quad G_X(s) = \sum_{x=0}^{\infty} \mathrm{P}(X=x)s^x,$

$\mathrm{P}(X=x)$ is the coefficient of s^x in the power series expansion of the r.h.s.

Now

$$\begin{aligned} G_X(s) &= \frac{1}{M+1}(1-s^{M+1})(1+s+s^2+\cdots), \quad |s|<1 \\ &= \frac{1}{M+1}(1+s+s^2+\cdots-s^{M+1}-s^{M+2}-\cdots) \\ &= \frac{1}{M+1}(1+s+\cdots+s^M). \end{aligned}$$

So

$$\mathrm{P}(X=x) = \frac{1}{M+1}, \qquad x = 0,1,...,M \qquad \text{(discrete uniform distribution.)}$$

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(iii) The total sum of the scores is

$$S_N = X_1 + \cdots + X_N,$$

where X_i is the score in the i^{th} game and the $\{X_i\}$ are independent, identically distributed random variables; the random variable N is the value obtained from throwing the die. Then the PGF of S_N is $G_N(G_X(s))$, where $G_N(u)$ is the PGF of N and $G_X(s)$ is the PGF of (any) game score.

Now

$$\begin{aligned} G_N(u) &= \sum_{n=1}^{6} \mathrm{P}(N=n)u^n \\ &= \sum_{n=1}^{6} \tfrac{1}{6}u^n = \tfrac{1}{6}.\frac{u(1-u^6)}{1-u}; \\ G_X(s) &= \sum_{x=0}^{2} \mathrm{P}(X=x)s^x \\ &= \tfrac{1}{10} + \tfrac{6}{10}s + \tfrac{3}{10}s^2. \end{aligned}$$

So the PGF of S_N is

$$\frac{1}{6}.\frac{\{\frac{1}{10}+\frac{6}{10}s+\frac{3}{10}s^2\}\{1-(\frac{1}{10}+\frac{6}{10}s+\frac{3}{10}s^2)^6\}}{1-(\frac{1}{10}+\frac{6}{10}s+\frac{3}{10}s^2)}.$$

We have that $\mathrm{E}(S_N) = \mathrm{E}(N)\mathrm{E}(X)$.

But

$$\begin{aligned} \mathrm{E}(N) &= \sum_{n=1}^{6} n.\mathrm{P}(N=n) = \tfrac{1}{6}\sum_{n=1}^{6} n = \tfrac{7}{2}; \\ \mathrm{E}(X) &= \left[G_X^{(1)}(s)\right]_{s=1} = \tfrac{12}{10}. \end{aligned}$$

So $\mathrm{E}(S_N) = \frac{7}{2} \times \frac{12}{10} = 4.2$.

3. (i) We have

$$\begin{aligned} G_X(s) &= \sum_{x=1}^{\infty} \mathrm{P}(X=x)s^x = \sum_{x=1}^{\infty} pq^{x-1}s^x \\ &= ps\sum_{x=1}^{\infty}(qs)^{x-1} = ps\sum_{r=0}^{\infty}(qs)^r \qquad [r=x-1] \\ &= \frac{ps}{1-qs}, \qquad |qs|<1. \end{aligned}$$

$$\begin{aligned} G_X^{(1)}(s) &= \frac{p}{1-qs} - \frac{ps(-q)}{(1-qs)^2}, \\ G_X^{(2)}(s) &= -\frac{p(-q)}{(1-qs)^2} + \frac{pq}{(1-qs)^2} - \frac{2pqs(-q)}{(1-qs)^3}. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{E}(X) &= G_X^{(1)}(1) = \frac{p}{1-q} + \frac{pq}{(1-q)^2} = 1 + \frac{q}{p} = \frac{1}{p}; \\ \mathrm{E}[X(X-1)] &= G_X^{(2)}(1) = \frac{2pq}{(1-q)^2} + \frac{2pq^2}{(1-q)^3} \\ &= \frac{2q}{p} + \frac{2q^2}{p^2} = \frac{2pq+2q^2}{p^2} = \frac{2q}{p^2}. \end{aligned}$$

So $\mathrm{Var}(X) = \dfrac{2q}{p^2} + \dfrac{1}{p} - \dfrac{1}{p^2} = \dfrac{2q+p-1}{p^2} = \dfrac{q}{p^2}.$

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(ii) (a) A sequence of successes and failures ending with the r^{th} success may be divided into r sub-sequences each consisting of a number of failures followed by a single success, e.g.

$$\underset{1}{\text{F F ... F S}} \,\big|\, \underset{2}{\text{F ... F S}} \,\big|\, \underset{3}{\text{S}} \,\big|\, \underset{4}{\text{F ... F S}} \,\big|\, \text{....} \,\big|\, \underset{r}{\text{F ... F S}} \,\big|$$

Let X_i be the number of trials in the i^{th} sub-sequence. Then $X_1, ..., X_r$ are independent random variables, each distributed with the geometric distribution defined in part (i). If Z denotes the number of trials required for r successes to occur,

$$Z = X_1 + X_2 + \cdots + X_r.$$

(b) Since $X_1, ..., X_r$ are independent random variables,

$$G_Z(s) = G_{X_1}(s)...G_{X_r}(s) = \left\{\frac{ps}{1-qs}\right\}^r, \quad |qs| < 1.$$

$\text{P}(Z = z)$ is the coefficient of s^z in the power series expansion of $G_Z(s)$, where $z = r, r+1, ...$ Using the binomial expansion for negative integer index (see Appendix to Lecture Notes), we have

$$G_Z(s) = p^r s^r \sum_{i=0}^{\infty} \binom{i+r-1}{i} (qs)^i, \qquad |qs| < 1.$$

Let $z = r + j$, where $j = 0, 1, ...$ Then the coefficient of $s^z = s^{r+j}$ in $G_Z(s)$ is

$$p^r \binom{j+r-1}{j} q^j = \binom{z-1}{z-r} p^r q^{z-r}, \qquad z = r, r+1, ...$$

i.e. $$\text{P}(Z = z) = \binom{z-1}{r-1} p^r q^{z-r}, \qquad z = r, r+1, ...$$

(c) $Z = \sum_{i=1}^{r} X_i$, so $\text{E}(Z) = \sum_{i=1}^{r} \text{E}(X_i) = \frac{r}{p}$.

Since the X_i's are independent,

$$\text{Var}(Z) = \sum_{i=1}^{r} \text{Var}(X_i) = \frac{rq}{p^2}.$$

Alternatively:

Determine $G_Z^{(1)}(s)$; then $\text{E}(X) = G_Z^{(1)}(1) = \frac{r}{p}$.

Determine $G_Z^{(2)}(s)$; then $\text{E}[X(X-1)] = G_Z^{(2)}(1) = \frac{r(r-1)}{p^2} + \frac{2rq}{p^2}$.

So

$$\begin{aligned} \text{Var}(Z) &= \frac{r(r-1)}{p^2} + \frac{2rq}{p^2} + \frac{r}{p} - \frac{r^2}{p^2} \\ &= \frac{r}{p}\left(-\frac{1}{p} + \frac{2q}{p} + 1\right) \qquad [-1 + 2q + p = q] \\ &= \frac{rq}{p^2}. \end{aligned}$$

4. (a) Since $X_0 = 1,\ \mathrm{P}(X_0 = k) = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1. \end{cases}$

So $\quad G_0(s) = \sum_k \mathrm{P}(X_0 = k)s^k = s.$

The PGF of the random variable C is

$$\begin{aligned} G(s) &= \sum_{k=0}^{\infty} \mathrm{P}(C = k)s^k \\ &= \sum_{k=0}^{\infty} (\tfrac{1}{2})^{k+1}s^k = \tfrac{1}{2}\sum_{k=0}^{\infty} \left(\tfrac{1}{2}s\right)^k \\ &= \tfrac{1}{2}.\frac{1}{1-\frac{1}{2}s}, \qquad |\tfrac{1}{2}s| < 1 \\ &= \frac{1}{2-s}, \qquad |s| < 2. \end{aligned}$$

So $\quad G_1(s) = G(s) = \dfrac{1}{2-s}, \qquad |s| < 2.$

Then

$$\begin{aligned} G_2(s) &= G_1(G(s)) = \frac{1}{2-\left(\frac{1}{2-s}\right)} = \frac{2-s}{3-2s} \\ G_3(s) &= G_2(G(s)) = \frac{2-\left(\frac{1}{2-s}\right)}{3-2\left(\frac{1}{2-s}\right)} = \frac{3-2s}{4-3s}. \end{aligned}$$

(b) The result

$$G_n(s) = \frac{n-(n-1)s}{(n+1)-ns}$$

holds for $n = 1$. Suppose it holds for $n = m$, i.e. that $G_m(s) = \dfrac{m-(m-1)s}{(m+1)-ms}$.

Then

$$\begin{aligned} G_{m+1}(s) &= G_m(G(s)) \\ &= \frac{m-(m-1)(\frac{1}{2-s})}{(m+1)-m(\frac{1}{2-s})} \\ &= \frac{2m-ms-m+1}{2m+2-ms-s-m} = \frac{(m+1)-ms}{(m+2)-(m+1)s} \end{aligned}$$

i.e. the result holds for $n = m+1$. So, by induction, it holds for all $n \geqslant 1$.

(c) $\mathrm{P}(X_n = 0)$ is the constant term in the power series expansion of $G_n(s)$; $\mathrm{P}(X_n = x), x \geqslant 1$ is the coefficient of s^x. Now

$$\begin{aligned} \frac{n-(n-1)s}{(n+1)-ns} &= \frac{n-(n-1)s}{(n+1)\{1-\frac{n}{n+1}s\}} \\ &= \tfrac{1}{n+1}\{n-(n-1)s\}\{1+(\tfrac{n}{n+1})s+(\tfrac{n}{n+1})^2s^2+\cdots\} \qquad |\tfrac{n}{n+1}s| < 1. \end{aligned}$$

Hence $\quad \mathrm{P}(X_0 = 0) = \frac{n}{n+1} \longrightarrow 1$ as $n \longrightarrow \infty$ i.e. ultimate extinction is certain.

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$$\begin{aligned}
\mathrm{P}(X_n = x) &= \left(\tfrac{n}{n+1}\right)\left(\tfrac{n}{n+1}\right)^x - \left(\tfrac{n-1}{n+1}\right)\left(\tfrac{n}{n+1}\right)^{x-1} \qquad x \geqslant 1 \\
&= \left(\tfrac{1}{n+1}\right)\left(\tfrac{n}{n+1}\right)^{x-1}\left\{n\left(\tfrac{n}{n+1}\right) - (n-1)\right\} \\
&= \left(\tfrac{1}{n+1}\right)\left(\tfrac{n}{n+1}\right)^{x-1}\left(\tfrac{1}{n+1}\right) \\
&= \frac{n^{x-1}}{(n+1)^{x+1}}, \qquad x = 1, 2, ...
\end{aligned}$$

5. (i) We have

$$\mathrm{E}(I_A) = \mathrm{P}(A), \quad \mathrm{E}(I_B) = \mathrm{P}(B),$$
$$\mathrm{E}(I_A I_B) = \mathrm{P}(A \cap B).$$

So

$$\begin{aligned}
\mathrm{Cov}(I_A, I_B) &= \mathrm{E}(I_A I_B) - \mathrm{E}(I_A)\mathrm{E}(I_B) \\
&= \mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B) \\
&= \mathrm{P}(A|B)\mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B) \\
&= \mathrm{P}(B)[\mathrm{P}(A|B) - \mathrm{P}(A)].
\end{aligned}$$

The result follows immediately.

(ii) Let I_i be the indicator random variable for A_i $(i = 1, ..., n)$.

Then $\quad X = \sum_{i=1}^{n} I_i$

and $\quad \mathrm{E}(X) = \sum_{i=1}^{n} \mathrm{E}(I_i) = \sum_{i=1}^{n} \mathrm{P}(A_i).$

But $X \geqslant Y$, so

$$\mathrm{E}(X) \geqslant \mathrm{E}(Y) = \mathrm{P}(Y = 1) = \mathrm{P}(X \geqslant 1) = \mathrm{P}(A_1 \cup \cdots \cup A_n),$$

proving the result.

(iii) Combine the outcomes E_i and E_j into one outcome E_{ij} (with probability $p_i + p_j$): we now have a multinomial situation with $k - 1$ outcomes (and n trials), so the distribution of $X_{ij} = X_i + X_j$ is binomial with variance $n(p_i + p_j)(1 - (p_i + p_j))$. Using the quoted formula we have:

$$\begin{aligned}
n(p_i + p_j)(1 - p_i - p_j) &= \mathrm{Var}(X_i) + \mathrm{Var}(X_j) + 2\mathrm{Cov}(X_i, X_j) \\
&= np_i(1 - p_i) + np_j(1 - p_j) + 2\mathrm{Cov}(X_i, X_j)
\end{aligned}$$

which yields $\quad \mathrm{Cov}(X_i, X_j) = -np_i p_j \quad$ as before.

6. We have $\quad X = 1 + \sum_{i=1}^{B} I_i$

so $\quad \mathrm{E}(X) = 1 + \sum_{i=1}^{B} \mathrm{E}(I_i) = 1 + \sum_{i=1}^{B} \mathrm{P}(I_i = 1).$

But $\quad \mathrm{P}(I_i) = 1) = \dfrac{1}{W + 1},$

since each ball from the set $\{$black ball i, all W white balls$\}$ has the same probability of being drawn.

So $\quad \mathrm{E}(X) = 1 + \dfrac{B}{W + 1}.$

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[*Comment* A question such as

> If cards are drawn at random from a standard pack, one by one, how many cards would one expect to draw before getting (a) a king; (b) a club; ....?

is just a special case of the above problem.]

7. Let $X_i =$ score on the i^{th} roll. Then

$$G_i(s) = \frac{1}{6}\sum_{x=1}^{6} s^x = \frac{5}{6}\sum_{y=0}^{5} s^y = \frac{s(1-s^6)}{6(1-s)}, \quad i = 1, 2, 3.$$

Since $X = \sum_{i=1}^{3} X_i$, $G_X(s) = \dfrac{s^3(1-s^6)^3}{6^3(1-s)^3}$. Then P$(X = 14)$ is the coefficient of s^{14} in the series expansion of $G_X(s)$, i.e. the coefficient of s^{11} in the expansion of

$$\frac{(1-s^6)^3}{6^3(1-s)^3} = \frac{1}{6}\left[1 - 3s^6 + 3s^{12} - s^{18}\right]\left[1 + \binom{3}{1}s + \binom{4}{2}s^2 + ...\right],$$

i.e.

$$\frac{1}{6^3}\left[\binom{13}{11} - 3\binom{7}{5}\right] = \frac{1}{6^3}\left[\frac{13\times 12}{12} - \frac{3\times 7\times 6}{2}\right] = \frac{5}{72}.$$

8. The PGF of each X_i is

$$G(s) = \sum_{k=1}^{\infty} \frac{(\frac{4}{5})^k s^k}{k\log_e 5} = -\frac{\log_e(1-\frac{4}{5}s)}{\log_e 5}.$$

Now $T = \sum_{i=1}^{N} X_i$, so $G_T(s) = G_N(G(s))$.

But $G_N(s) = e^{\lambda(s-1)} = e^{\log_e 5(s-1)}$.

So

$$\begin{aligned} G_T(s) &= \exp\left[\log_e 5\left\{-\frac{\log_e(1-\frac{4}{5}s)}{\log_e 5} - 1\right\}\right] \\ &= \exp\left[-\log_e(1-\tfrac{4}{5}s)\right]\exp[-\log_e 5] \\ &= \frac{\frac{1}{5}}{1-\frac{4}{5}s}. \end{aligned}$$

But the PGF of the modified geometric distribution is

$$\sum_{k=0}^{\infty} pq^k s^k = p\sum_{k=0}^{\infty}(qs)^k = \frac{p}{1-qs}, \quad |qs| < 1.$$

So T has the modified geometric distribution with parameter $p = \frac{1}{5}$.