SOR201 Solutions to Examples 5

1. (i) The possible states are: 0,1,2,3.

The times are: $n = 0, 1, 2, \ldots$

$X_{n-1} = i \to X_n = i$ Conditional prob. $i/3$ $\quad 1 \leqslant i \leqslant 3$

$X_{n-1} = i \to X_n = i+1$ Conditional prob. $(3-i)/3$ $\quad 0 \leqslant i \leqslant 2.$

Other transitions have zero probabilities. The state of the system at time n depends on the state at time $(n-1)$ but not on the states at times $0, 1, \ldots, (n-2)$. Hence $\{X_n\}$is a Markov chain: it is also homogeneous, since the transition probabilities are not functions of n.

The transition probability matrix is

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Let $\boldsymbol{p}^{(n)}$ denote the row vector of absolute probabilities at time n, i.e.

$$\boldsymbol{p}^{(n)} = (\mathrm{P}(X_n = 0), \mathrm{P}(X_n = 1), \mathrm{P}(X_n = 2), \mathrm{P}(X_n = 3)).$$

Then

$$\begin{aligned} \boldsymbol{p}^{(2)} = \boldsymbol{p}^{(0)} \boldsymbol{P}^2 &= (1, 0, 0, 0) \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{9} & \frac{6}{9} & \frac{2}{9} \\ 0 & 0 & \frac{4}{9} & \frac{5}{9} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \left(0, \tfrac{1}{3}, \tfrac{2}{3}, 0\right). \end{aligned}$$

(ii) $X_n = Y_1 + Y_2 + \cdots + Y_n = X_{n-1} + Y_n$.

Given $X_{n-1} = i$, then $X_n = i + k = j$ with probability a_k, so we only need to know the state at time $(n-1)$ to make a conditional probability statement about X_n. Hence $\{X_n\}$ is a Markov chain: it is also homogeneous, since the transition probabilities are not functions of n. We have

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \ldots & \ldots \end{array} \\ \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots & \ldots \\ 0 & 0 & a_0 & a_1 & \ldots & \ldots \\ 0 & 0 & 0 & a_0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \end{array}.$$

/continued overleaf

(iii) Ehrenfest model for diffusion

$X_{n-1} = i \nearrow X_n = i+1$ if the randomly selected molecule comes from urn B: conditional probability is $(M-i)/M, \quad 0 \leqslant i \leqslant M-1$

$X_{n-1} = i \searrow X_n = i-1$ if the randomly selected molecule comes from urn A: conditional probability is $i/M, \quad 1 \leqslant i \leqslant M.$

Other transitions have zero probabilities. Once again, we only need to know X_{n-1} to make a conditional probability statement about X_n, and the transition probabilities are not functions of n. So $\{X_n\}$ is a homogeneous Markov chain, with

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M{-}1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & M{-}2 & M{-}1 & M \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{M} & 0 & \frac{M-1}{M} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{2}{M} & 0 & \frac{M-2}{M} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{M-1}{M} & 0 & \frac{1}{M} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

2. (i) (a) Since the transition probabilities are homogeneous, we have

$$\begin{array}{lcll} \mathrm{P}(X_n = 1 | X_{n-1} = 0) & = & p_{01} & \text{– element (0,1) in } \boldsymbol{P} \\ \mathrm{P}(X_m = 0 | X_{m-2} = 1) & = & p_{10}^{(2)} & \text{– element (1,0) in } \boldsymbol{P}^2 \\ \mathrm{P}(X_{r+3} = 1 | X_r = 1) & = & p_{11}^{(3)} & \text{– element (1,1) in } \boldsymbol{P}^3. \end{array}$$

Now

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}.$$

So $\mathrm{P}(X_n = 1 | X_{n-1} = 0) = \frac{2}{3}$.

$$\boldsymbol{P}^2 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{4}{9} & \frac{5}{9} \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix}.$$

So $\mathrm{P}(X_m = 0 | X_{m-2} = 1) = \frac{5}{12}$.

$$\boldsymbol{P}^3 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{9} & \frac{5}{9} \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix} = \begin{pmatrix} \frac{23}{54} & \frac{31}{54} \\ \frac{31}{72} & \frac{41}{72} \end{pmatrix}.$$

So $\mathrm{P}(X_{r+3} = 1 | X_r = 1) = \frac{41}{72}$.

(b) $\boldsymbol{p}^{(n)} = \boldsymbol{p}^{(0)} \boldsymbol{P}^n$, where $\boldsymbol{p}^{(n)} = (\mathrm{P}(X_n = 0), \mathrm{P}(X_n = 1))$.
Initially, the system is equally likely to be in state 0 or state 1: this means that

$$\boldsymbol{p}^{(0)} = \left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

Then

$$\boldsymbol{p}^{(1)} = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\tfrac{5}{12}, \tfrac{7}{12}\right).$$

/continued overleaf

So $\qquad \mathrm{P}(X_1 = 1) = \frac{7}{12} \approx 0.583.$

$$\boldsymbol{p}^{(2)} = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \begin{pmatrix} \frac{4}{9} & \frac{5}{9} \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix} = \left(\tfrac{31}{72}, \tfrac{41}{72}\right).$$

So $\qquad \mathrm{P}(X_2 = 1) = \frac{41}{72} \approx 0.569.$

$$\boldsymbol{p}^{(3)} = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \begin{pmatrix} \frac{23}{54} & \frac{31}{54} \\ \frac{31}{72} & \frac{41}{72} \end{pmatrix} = \left(\tfrac{185}{432}, \tfrac{247}{432}\right).$$

So $\qquad \mathrm{P}(X_3 = 1) = \frac{247}{432} \approx 0.572.$

(c) The given Markov chain is finite, aperiodic and irreducible (states 0 and 1 form a closed set). Hence we can use Markov's theorem to calculate $\lim_{n\to\infty} \boldsymbol{P}^n$. This limiting matrix will be an approximation to $\boldsymbol{P}^n$ when n is large. Thus

$$\lim_{n\to\infty} \boldsymbol{P}^n = \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix},$$

where π_0, π_1 satisfy

$$(\pi_0, \pi_1) = (\pi_0, \pi_1) \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
$$\pi_0 + \pi_1 = 1, \quad \pi_0 > 0, \pi_1 > 0.$$

i.e.

$$\begin{array}{lllll} \pi_0 = \frac{1}{3}\pi_0 + \frac{1}{2}\pi_1 & \rightarrow & \frac{2}{3}\pi_0 = \frac{1}{2}\pi_1 & \rightarrow & \pi_1 = \frac{4}{3}\pi_0 \\ \pi_1 = \frac{2}{3}\pi_0 + \frac{1}{2}\pi_1 & \rightarrow & \frac{2}{3}\pi_0 = \frac{1}{2}\pi_1 & \rightarrow & \pi_1 = \frac{4}{3}\pi_0 \end{array}$$

(note that one equation is *redundant*).

Normalizing: $\quad \pi_0 + \frac{4}{3}\pi_0 = 1 \ \rightarrow \ \pi_0 = \frac{3}{7} \ \rightarrow \ \pi_1 = \frac{4}{7}.$

So

$$\lim_{n\to\infty} \boldsymbol{P}^n = \begin{pmatrix} \frac{3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{4}{7} \end{pmatrix}.$$

Hence $\qquad \mathrm{P}(X_n = 1) \approx 0.571 \qquad$ when n is large.

(ii) We have

$$\begin{aligned} \boldsymbol{P}^2 &= \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{5}{6} & 0 \\ \frac{2}{9} & \frac{11}{18} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} \\ \boldsymbol{p}^{(0)} &= (1, 0, 0). \end{aligned}$$

Then

(a)

$$\begin{aligned} \mathrm{P}(X_0 = 0, X_1 = 1, X_2 = 1) &= \mathrm{P}(X_2 = 1 | X_0 = 0, X_1 = 1).\mathrm{P}(X_0 = 0, X_1 = 1) \\ &= \mathrm{P}(X_2 = 1 | X_1 = 1).\mathrm{P}(X_1 = 1 | X_0 = 0).\mathrm{P}(X_0 = 0) \\ &\qquad \text{[using the Markov property in the first term]} \\ &= p_{11}.p_{01}.\mathrm{P}(X_0 = 0) \\ &= \tfrac{2}{3}.\tfrac{1}{2}.1 = \tfrac{1}{3}. \end{aligned}$$

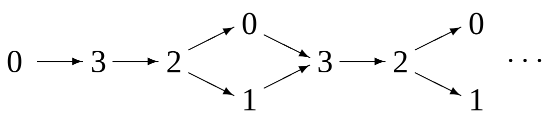
(b) $\mathrm{P}(X_n = 1 | X_{n-2} = 0) = p_{01}^{(2)} = \frac{5}{6}.$

(c) $(\mathrm{P}(X_2 = 0), \mathrm{P}(X_2 = 1), \mathrm{P}(X_2 = 2)) = \boldsymbol{p}^{(2)}$
$\qquad = p^{(0)}\boldsymbol{P}^2 = (1, 0, 0)\boldsymbol{P}^2 = \left(\frac{1}{6}, \frac{5}{6}, 0\right).$

3. (i) (a) The $\boldsymbol{P}$ matrix and possible transitions are:

$$\begin{array}{c} \\ 0\\1\\2\\3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{l} 0 \to 3 \\ 1 \to 3 \\ 2 \to 0,1 \\ 3 \to 2. \end{array}$$

The chain is irreducible, implying that all states are recurrent.



$$p_{00}^{(1)} = 0,\ p_{00}^{(2)} = 0,\ p_{00}^{(3)} > 0,\ p_{00}^{(4)} = 0,\ p_{00}^{(5)} = 0,\ p_{00}^{(6)} > 0,\ \ldots$$

So state 0 has period 3: hence all states are *periodic* with period 3.

(b) The $\boldsymbol{P}$ matrix and possible transitions are:

$$\begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array} \qquad \begin{array}{l} 0 \to 0,2 \\ 1 \to 0,1,2 \\ 2 \to 0,2 \\ 3 \to 3,4 \\ 4 \to 3,4. \end{array}$$

$\{3,4\}$ is an irreducible closed set: its states are *recurrent* and *aperiodic*.
Similarly for $\{0,2\}$.
State 1 is *transient* and *aperiodic*.

(c) The $\boldsymbol{P}$ matrix and possible transitions are:

$$\begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{l} 0 \to 1 \\ 1 \to 2 \\ 2 \to 0 \\ 3 \to 3 \\ 4 \to 4. \end{array}$$

States 3 and 4 are *absorbing*.
$\{0,1,2\}$ is an irreducible closed set: its states are *recurrent* with *period 3*.

(d) The $\boldsymbol{P}$ matrix and possible transitions are:

$$\begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{l} 0 \to 0,1 \\ 1 \to 0,1 \\ 2 \to 2 \\ 3 \to 2,3 \\ 4 \to 0. \end{array}$$

/continued overleaf

$\{0, 1\}$ is an irreducible closed set with *recurrent*, *aperiodic* states.
State 2 is *absorbing*.
States 3 and 4 are *transient*, *aperiodic* states.

(ii) The $\boldsymbol{P}$ matrix and possible transitions are:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \end{array} \qquad \begin{array}{l} 0 \to 1 \\ 1 \to 0, 2 \\ 2 \to 0, 1, 2. \end{array}$$

This Markov chain is finite, aperiodic and irreducible. So by Markov's theorem,
$\boldsymbol{P}^n \to \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \\ \pi_0 & \pi_1 & \pi_2 \\ \pi_0 & \pi_1 & \pi_2 \end{pmatrix}$ as $n \to \infty$,
where $(\pi_0, \pi_1, \pi_2) = (\pi_0, \pi_1, \pi_2)\boldsymbol{P}$ and $\pi_0 + \pi_1 + \pi_2 = 1, \quad \pi_0, \pi_1, \pi_2 > 0,$ i.e.

$$\left.\begin{array}{rcl} \pi_0 &=& \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 \\ \pi_1 &=& \pi_0 + \frac{1}{4}\pi_2 \\ \pi_2 &=& \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 \end{array}\right\} \text{one of these is redundant.}$$

The normalized solution is $\pi_0 = \frac{5}{15}, \ \pi_1 = \frac{6}{15}, \ \pi_2 = \frac{4}{15}$.

4. (a) $\underline{r = 1}$

$X_{n-1} = 0 \to X_n = 0$ Conditional prob. q
$X_{n-1} = 0 \to X_n = 1$ Conditional prob. p

$X_{n-1} = 1 \to X_n = 0$ Conditional prob. 0
$X_{n-1} = 1 \to X_n = 1$ Conditional prob. $1.$

$\{X_n\}$ is a Markov chain since the state at time n is influenced only by the state at time $n-1$, not by the states at earlier times. The transition probabilities are not functions of n, so the chain is homogeneous.

$$\boldsymbol{P} = \begin{pmatrix} q & p \\ 0 & 1 \end{pmatrix}.$$

$\underline{r > 1}$

$X_{n-1} = i \to X_n = i$ Conditional prob. q
$X_{n-1} = i \to X_n = i + 1$ Conditional prob. p
for $i = 0, ..., r - 1$

$X_{n-1} = r \longrightarrow X_n = r$ Conditional prob. $1.$

/continued overleaf

Other transition probabilities are zero. For the reasons given above, $\{X_n\}$ is again a homogeneous Markov chain.

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ r-1 \\ r \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & r-1 & r \end{array} \\ \begin{pmatrix} q & p & 0 & 0 & \cdots & 0 & 0 \\ 0 & q & p & 0 & \cdots & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \end{array}.$$

(b) $\underline{r = 1}$ State 1 is *absorbing*, state 0 is *transient*. Let f_{01} be the probability of eventually entering state 1, starting from state 0, i.e. the probability of absorption. Then

$$f_{01} = p_{01} + p_{00} f_{01} = p + q f_{01}.$$

Hence $f_{01} = 1$, i.e. absorption is certain.
Let μ_0 denote the mean time to absorption. Then

$$\mu_0 = 1 + p_{00}\mu_0 = 1 + q\mu_0.$$

Hence $\mu_0 = \dfrac{1}{1-q} = \dfrac{1}{p}$ (cf. geometric distribution).

(c) $\underline{r > 1}$ State r is *absorbing*, states $0, 1, ..., (r-1)$ are *transient*.
Let $T = \{0, 1, ..., (r-1)\}$. Let f_{ir} be the probability of eventual absorption in state r, starting from state $i, i \in T$. Then

$$f_{ir} = p_{ir} + \sum_{j \in T} p_{ij} f_{jr}, \qquad i \in T.$$

Now

$$\begin{array}{lll} p_{r-1,r} = p, & \text{otherwise } p_{ir} = 0 & \text{for } i \in T \\ p_{i,i} = q, p_{i,i+1} = p, & \text{otherwise } p_{ij} = 0 & \text{for } i, j, \in T. \end{array}$$

So

$$\begin{array}{rcl} f_{0r} & = & q f_{0r} + p f_{1r} \\ f_{1r} & = & q f_{1r} + p f_{2r} \\ & \cdots\cdots & \\ f_{r-2,r} & = & q f_{r-2,r} + p f_{r-1,r} \\ f_{r-1,r} & = & p + q f_{r-1,r}. \end{array}$$

[Solution(not required): working backwards, $f_{r-1,r} = 1, f_{r-2,r} = 1, ..., f_{0,r} = 1$.]
Let μ_i be the mean time to absorption, starting from state i. Then

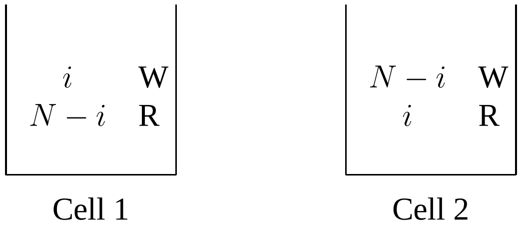
$$\mu_i = 1 + \sum_{j \in T} p_{ij}\mu_j, \qquad i \in T$$

i.e.

$$\begin{array}{rcl} \mu_0 & = & 1 + q\mu_0 + p\mu_1 \\ \mu_1 & = & 1 + q\mu_1 + p\mu_2 \\ & \cdots\cdots & \\ \mu_{r-2} & = & 1 + q\mu_{r-2} + p\mu_{r-1} \\ \mu_{r-1} & = & 1 + q\mu_{r-1} \end{array}$$

[Solution (not required): Working backwards, $\mu_{r-1} = \dfrac{1}{p},\ \mu_{r-2} = \dfrac{2}{p},\ ...,\ \mu_0 = \dfrac{r}{p}$.
Usually the system would be starting in state 0.]

5. The system can be represented thus:



The possible transitions are:

$X_{n-1} = i \to X_n = i+1$: if R from 1 and W from 2:
for $0 \leqslant i \leqslant N-1$, cond. prob. is $\left(\frac{N-i}{N}\right)^2$

$X_{n-1} = i \to X_n = i$: if (W from 1 and W from 2) *OR* (R from 1 and R from 2):
for $1 \leqslant i \leqslant N-1$, cond. prob. is $2\left(\frac{i}{N}\right)\left(\frac{N-i}{N}\right)$

$X_{n-1} = i \to X_n = i-1$: if W from 1 and R from 2:
for $1 \leqslant i \leqslant N$, cond. prob. is $\left(\frac{i}{N}\right)^2$

All other transition probabilities are zero.

We have a Markov chain because we only require to know the state after step $n-1$ in order to make a conditional probability statement about the state of the system after step n. The chain is homogeneous since the transition probabilities are not functions of n. The $\boldsymbol{P}$ matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & N-2 & N-1 & N \end{array} \\
\left(\begin{array}{cccccccc}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\left(\frac{1}{N}\right)^2 & 2\left(\frac{1}{N}\right)\left(\frac{N-i}{N}\right) & \left(\frac{N-1}{N}\right)^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & \left(\frac{2}{N}\right)^2 & 2\left(\frac{2}{N}\right)\left(\frac{N-2}{N}\right) & \left(\frac{N-2}{N}\right)^2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \left(\frac{N-1}{N}\right)^2 & 2\left(\frac{1}{N}\right)\left(\frac{N-1}{N}\right) & \left(\frac{1}{N}\right)^2 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{array}\right)
\end{array}.$$

Clearly the distribution of X_0 is hypergeometric, viz.

$$\mathrm{P}(X_0 = i) = \binom{N}{i}\binom{N}{N-i} \Big/ \binom{2N}{N}.$$

Then $\boldsymbol{p}^{(n)} = \boldsymbol{p}^{(0)}\boldsymbol{P}^n$, where $\boldsymbol{p}^{(r)} = (\mathrm{P}(X_r = 0), ..., \mathrm{P}(X_r = N))$.

6. We have:

$$\boldsymbol{P}^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \\ \frac{1}{16} & \frac{7}{16} & \frac{1}{2} \\ \frac{5}{16} & \frac{1}{4} & \frac{7}{16} \end{pmatrix}.$$

Then

$$(\mathrm{P}(X_2 = 0), \mathrm{P}(X_2 = 1), \mathrm{P}(X_2 = 2)) = \boldsymbol{p}^{(2)} = \boldsymbol{p}^{(0)} \boldsymbol{P}^2 = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) \begin{pmatrix} \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \\ \frac{1}{16} & \frac{7}{16} & \frac{1}{2} \\ \frac{5}{16} & \frac{1}{4} & \frac{7}{16} \end{pmatrix}.$$

So

$$\begin{aligned} \mathrm{P}(X_2 = 1) &= \tfrac{1}{3}\left(\tfrac{1}{8} + \tfrac{7}{16} + \tfrac{1}{4}\right) &= \tfrac{13}{48} \\ \mathrm{P}(X_2 = 2) &= \tfrac{1}{3}\left(\tfrac{3}{8} + \tfrac{1}{2} + \tfrac{7}{16}\right) &= \tfrac{7}{16}. \end{aligned}$$

By Markov's theorem, a limiting distribution $\boldsymbol{\pi}$ exits because the chain is finite, aperiodic and irreducible. $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi} = \boldsymbol{\pi P}$, with $\sum_i \pi_i = 1$. Thus

$$\begin{aligned} \pi_0 &= \phantom{\tfrac{1}{2}\pi_0 +} \tfrac{3}{4}\pi_1 + \tfrac{1}{4}\pi_2 && (1) \\ \pi_1 &= \tfrac{1}{2}\pi_0 \phantom{+ \tfrac{1}{4}\pi_1} + \tfrac{1}{4}\pi_2 && (2) \\ \pi_2 &= \tfrac{1}{2}\pi_0 + \tfrac{1}{4}\pi_1 + \tfrac{1}{2}\pi_2 && (3) \\ \text{and} \quad & \pi_0 + \pi_1 + \pi_2 = 1. \end{aligned}$$

Regard (3) as redundant. Then from (1) and (2) (by subtraction)

$\pi_0 - \pi_1 = -\frac{1}{2}\pi_0 + \frac{3}{4}\pi_1$, i.e $\frac{3}{2}\pi_0 = \frac{7}{4}\pi_1$, i.e. $\pi_1 = \frac{6}{7}\pi_0$.

Then from (2): $\pi_2 = \frac{10}{7}\pi_0$.

π_0 is found from the normalization requirement

$$\pi_0 + \pi_1 + \pi_2 = 1 = \pi_0 + \tfrac{6}{7}\pi_0 + \tfrac{10}{7}\pi_0.$$

This gives $\pi_0 = \frac{7}{23}$ and then $\pi_1 = \frac{6}{23}$, $\pi_2 = \frac{10}{23}$.

[Check: from (3), $\frac{10}{23} = \frac{7}{46} + \frac{3}{46} + \frac{10}{46} = \frac{10}{23}$. $\surd$]

7. States 1 and 3 are *absorbing*, while states 0, 2 and 4 are *transient*.

General form of equations for $\{f_{ik}\}$:

$$f_{ik} = p_{ik} + \sum_{j \in T} p_{ij} f_{jk}, \qquad i \in T.$$

In this case:

$\underline{k = 1}$

$$\begin{aligned} f_{01} &= \phantom{\tfrac{1}{3} +} \tfrac{1}{2}f_{01} + \tfrac{1}{4}f_{21} + \tfrac{1}{4}f_{41} && (1) \\ f_{21} &= \tfrac{1}{3} + \tfrac{1}{3}f_{01} && (2) \\ f_{41} &= \phantom{\tfrac{1}{3} +} \tfrac{1}{4}f_{01} + \tfrac{1}{4}f_{21} + \tfrac{1}{4}f_{41}. && (3) \end{aligned}$$

Substituting (2) in (1) and (3) we get

$$\begin{aligned} \tfrac{5}{12}f_{01} - \tfrac{1}{4}f_{41} &= \tfrac{1}{12} \\ \tfrac{1}{3}f_{01} - \tfrac{3}{4}f_{41} &= -\tfrac{1}{12}. \end{aligned}$$

Then $f_{01} = \frac{4}{11}$, $f_{41} = \frac{3}{11}$ and finally $f_{21} = \frac{5}{11}$.

/*continued overleaf*

$\underline{k=3}$

$$\begin{array}{rcrcrcrcrr} f_{03} & = & & & \frac{1}{2}f_{03} & + & \frac{1}{4}f_{23} & + & \frac{1}{4}f_{43} & \quad (4)\\ f_{23} & = & \frac{1}{3} & + & \frac{1}{3}f_{03} & & & & & \quad (5)\\ f_{43} & = & \frac{1}{4} & + & \frac{1}{4}f_{03} & + & \frac{1}{4}f_{23} & + & \frac{1}{4}f_{43}. & \quad (6) \end{array}$$

Substituting (5) in (4) and (6) we get

$$\begin{array}{rcrcr} \frac{5}{12}f_{03} & - & \frac{1}{4}f_{43} & = & \frac{1}{12}\\ \frac{3}{4}f_{43} & - & \frac{1}{3}f_{03} & = & \frac{1}{3} \end{array}$$

whence $f_{43}=\frac{8}{11},\ f_{03}=\frac{7}{11}$ and then $f_{23}=\frac{6}{11}$.

The general equations for the $\{\mu_i\}$ are

$$\mu_i = 1 + \sum_{j\in T} p_{ij}\mu_j, \qquad i \in T.$$

So here:

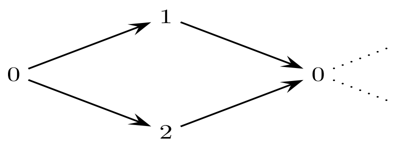
$$\begin{array}{rcrcrcrcr} \mu_0 & = & 1 & + & \frac{1}{2}\mu_0 & + & \frac{1}{4}\mu_2 & + & \frac{1}{4}\mu_4\\ \mu_2 & = & 1 & + & \frac{1}{3}\mu_0 & & & & \\ \mu_4 & = & 1 & + & \frac{1}{4}\mu_0 & + & \frac{1}{4}\mu_2 & + & \frac{1}{4}\mu_4. \end{array}$$

Proceeding as above, we deduce that

$$\mu_0=\frac{60}{11},\quad \mu_2=\frac{31}{11},\quad \mu_4=\frac{45}{11}.$$

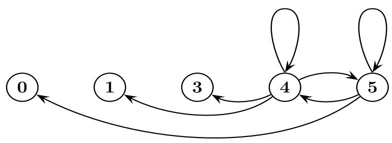
8.

$$\boldsymbol{P} = \left(\begin{array}{ccc|ccc} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & \boxed{1} & 0 & 0\\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{8}\\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{array}\right).$$



(0,1,2) - closed irreducible set of *periodic* states (period =2)

(3) - *absorbing* state



(4,5) - irreducible set of *transient, aperiodic* states.