SOR201 Solutions to Examples 6

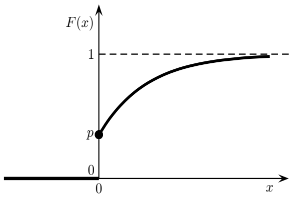
1. (i) The c.d.f. of the r.v. X is $F(x) = \mathrm{P}(X \leqslant x), \quad -\infty < x < \infty.$

(a) The event $(-\infty < X \leqslant x) \subseteq$ the event $(-\infty < X \leqslant y)$ when $x \leqslant y$.
So $\mathrm{P}(-\infty < X \leqslant x) \leqslant \mathrm{P}(-\infty < X \leqslant y), \qquad x \leqslant y$
i.e. $F(x) \leqslant F(y)$ when $x \leqslant y$.

(b) $F(-\infty) = \mathrm{P}(X \leqslant -\infty) = 0,$
$F(+\infty) = \mathrm{P}(X \leqslant +\infty) = 1.$

(c) $(-\infty < X \leqslant b) = (-\infty < X \leqslant a) \cup (a < X \leqslant b)$
– the union of 2 m.e. events.
So $\mathrm{P}(-\infty < X \leqslant b) = \mathrm{P}(-\infty < X \leqslant a) + \mathrm{P}(a < X \leqslant b)$
i.e $\mathrm{P}(a < X \leqslant b) = \mathrm{P}(-\infty < X \leqslant b) - \mathrm{P}(-\infty < X \leqslant a)$
$= F(b) - F(a).$

(ii) (a)



Discontinuity at $x = 0$.

(b) The c.d.f. is

$$F(x) = \begin{cases} 0, & x < 0 \\ p, & x = 0 \\ p + k(1 - e^{-\lambda x}) = p + (1-p)(1 - e^{-\lambda x}), & x > 0. \end{cases}$$

[k determined as $(1-p)$ by requiring that $F(x) \to 1$ as $x \to \infty$].

(iii) $f(x)$ is symmetrical about the point a, i.e.

$$f(a - y) = f(a + y), \qquad y \geqslant 0. \qquad (**)$$

Then

$$\int_{-\infty}^{a} f(x)dx = \int_{\infty}^{0} f(a+y)dy \qquad [\text{setting } x = a - y]$$
$$= \int_{0}^{\infty} f(a-y)dy$$

and

$$\int_{a}^{\infty} f(x)dx = \int_{0}^{\infty} f(a-y)(-1)dy \qquad [\text{setting } x = a + y]$$
$$= \int_{0}^{\infty} f(a-y)dy \qquad [\text{using } (**)]$$

So $\displaystyle\int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx.$

But $\displaystyle\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1.$

So

$$\int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx = \tfrac{1}{2}$$

i.e. $\mathrm{P}(X \leqslant a) = \mathrm{P}(X \geqslant a) = \frac{1}{2}$

i.e. the median of X is a.

/continued overleaf

Also

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{-\infty}^{a} xf(x)dx + \int_{a}^{\infty} xf(x)dx \\
&\qquad\qquad [\text{set } y = a - x \text{ in first integral, } y = x - a \text{ in second}] \\
&= \int_{\infty}^{0} (a-y)f(a-y)(-1)dy + \int_{0}^{\infty} (a+y)f(a+y)dy \\
&= \int_{0}^{\infty} (a-y)f(a-y)dy + \int_{0}^{\infty} (a+y)f(a-y)dy \qquad [\text{using } (**)] \\
&= \int_{0}^{\infty} 2af(a-y)dy \\
&\qquad\qquad [\text{use } \textstyle\int_0^\infty f(a-y)dy = \int_a^\infty f(x)dx = \frac{1}{2}] \\
&= 2a.\tfrac{1}{2} = a.
\end{aligned}$$

2. (i) The c.d.f. is

$$F(x) = \begin{cases} 1 - \exp(-\frac{1}{2}x^2), & x \geqslant 0 \\ 0, & x < 0. \end{cases}$$

So

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} x\exp(-\frac{1}{2}x^2), & x \geqslant 0 \\ 0, & x < 0. \end{cases}$$

Then

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{0}^{\infty} x^2 e^{-\frac{1}{2}x^2} dx \qquad [\text{set } t = \tfrac{1}{2}x^2,\ dt = xdx = \sqrt{2t}dx] \\
&= \int_{0}^{\infty} 2te^{-t}\frac{1}{\sqrt{2t}}dt \\
&= \sqrt{2}\int_{0}^{\infty} t^{\frac{3}{2}-1}e^{-t}dt \\
&= \sqrt{2}\Gamma(\tfrac{3}{2}) = \sqrt{2}.\tfrac{1}{2}.\Gamma(\tfrac{1}{2}) = \tfrac{1}{\sqrt{2}}.\sqrt{\pi} = \sqrt{\pi/2}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_{0}^{\infty} x^3 e^{-\frac{1}{2}x^2} dx \qquad [\text{set } t = \tfrac{1}{2}x^2] \\
&= \int_{0}^{\infty} (2t)^{\frac{3}{2}} e^{-t}\frac{1}{\sqrt{2t}}dt \\
&= 2\int_{0}^{\infty} te^{-t}dt = 2\Gamma(2) = 2.
\end{aligned}$$

So $\quad \mathrm{Var}X = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 2 - \pi/2.$

The median m of a continuous distribution is such that $F(m) = \frac{1}{2}$, so here

$$1 - \exp(-\tfrac{1}{2}m^2) = \tfrac{1}{2}$$

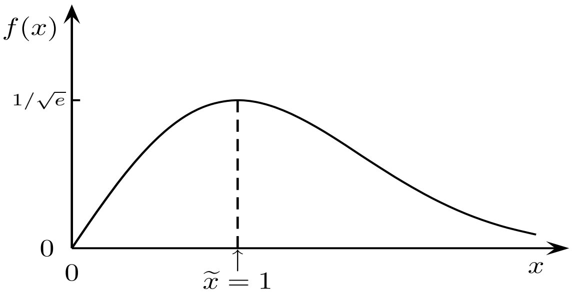
i.e. $\quad \exp(-\frac{1}{2}m^2) = \frac{1}{2}$ or $\exp(\frac{1}{2}m^2) = 2,$

so $\quad \frac{1}{2}m^2 = \log_e 2$

i.e. $\quad m = \{2\log_e 2\}^{\frac{1}{2}}.$

/continued overleaf

Also $\dfrac{df(x)}{dx} = e^{-\frac{1}{2}x^2} - x^2 e^{-\frac{1}{2}x^2} = 0$ when $x = \pm 1$. So mode $\widetilde{x} = 1$.



(ii)

$$f_X(x) = \begin{cases} kx^{p-1}/(1+x)^{p+q}, & x \geqslant 0;\ p, q > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The transformation $y = 1/(1+x),\ \ x \geqslant 0$ is one-to-one and differentiable:

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2} = -y^2.$$

The inverse transformation is

$$x = \frac{(1-y)}{y}, \qquad \frac{1-y}{y} \geqslant 0 \quad \text{or} \quad \underline{0 < y \leqslant 1.}$$

So

$$\begin{aligned} f_Y(y) &= f_X\left(\frac{1-y}{y}\right)\left|\frac{dx}{dy}\right| \\ &= \begin{cases} k\left(\dfrac{1-y}{y}\right)^{p-1} y^{p+q}\left|-\dfrac{1}{y^2}\right|, & 0 < y \leqslant 1 \\ 0, & \text{otherwise} \end{cases} \\ \text{i.e.}\quad f_Y(y) &= \begin{cases} ky^{q-1}(1-y)^{p-1}, & 0 < y \leqslant 1 \\ 0, & \text{otherwise} \end{cases} \\ \text{i.e.}\quad Y &\sim \text{beta}(q, p). \end{aligned}$$

(iii)

$$f_X(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

The transformation $y = \log_e\{1/(1-x)\},\ \ 0 \leqslant x \leqslant 1$ is one-to-one and differentiable. The inverse transformation is

$$\begin{aligned} x &= 1 - e^{-y}, \quad 0 \leqslant 1 - e^{-y} \leqslant 1, \text{ i.e. } 0 \leqslant e^{-y} \leqslant 1, \text{ i.e. } \underline{0 \leqslant y < \infty} \\ \text{and}\quad \frac{dx}{dy} &= e^{-y}. \end{aligned}$$

So

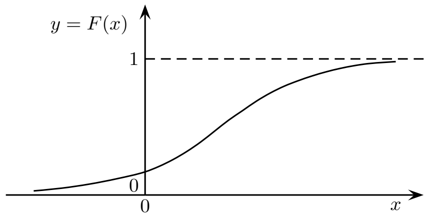
$$\begin{aligned} f_Y(y) &= f_X(1 - e^{-y})\left|\frac{dx}{dy}\right| \\ &= \begin{cases} e^{-y}, & 0 \leqslant y < \infty \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

– the negative exponential distribution with $\lambda = 1$.

/continued overleaf

(iv) The transformation $y = F_X(x), \quad -\infty < x < \infty$ is one-to-one and differentiable, and

$$\frac{dy}{dx} = f_X(x), \qquad 0 \leqslant y \leqslant 1.$$



So

$$f_Y(y) = f_X\left\{F_X^{-1}(y)\right\}\left|\frac{dx}{dy}\right| = f_X(x).\frac{1}{f_X(x)} = 1, \quad 0 \leqslant y \leqslant 1,$$

i.e. $Y \sim$ uniform$[0,1]$.

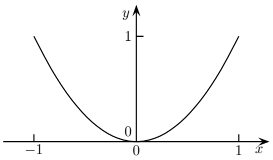
3. (i) (a) $f_X(x) = 2x\exp(-x^2), \quad 0 \leqslant x \leqslant \infty.$

The transformation $y = x^2, \; 0 \leqslant x < \infty$ is one-to-one and differentiable. Its inverse is $x = +\sqrt{y}$ and $\dfrac{dy}{dx} = 2x = 2\sqrt{y}, \quad 0 \leqslant y < \infty.$
So

$$\begin{aligned} f_Y(y) &= f_X(\sqrt{y})\left|\frac{dx}{dy}\right| \\ &= 2\sqrt{y}e^{-y}\left|\frac{1}{2\sqrt{y}}\right| = e^{-y}, \quad 0 \leqslant y < \infty \end{aligned}$$

(b)

$$f_X(x) = \begin{cases} \frac{1}{2}(1+x), & -1 \leqslant x \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$



The transformation $y = x^2$ is *not* one-to-one when $-1 \leqslant x \leqslant 1$. Then
EITHER

$$\begin{aligned} F_Y(y) &= \mathrm{P}(Y \leqslant y) = \mathrm{P}(X^2 \leqslant y) = \mathrm{P}(-\sqrt{y} \leqslant X \leqslant +\sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx = \int_{-\sqrt{y}}^{\sqrt{y}} \tfrac{1}{2}(1+x)dx, \qquad -1 \leqslant -\sqrt{y} \leqslant \sqrt{y} \leqslant 1 \\ &= \left[\tfrac{1}{2}x + \tfrac{1}{4}x^2\right]_{-\sqrt{y}}^{\sqrt{y}} = \sqrt{y}, \quad 0 \leqslant y \leqslant 1 \end{aligned}$$

So

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}, & 0 \leqslant y \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

OR
Divide the interval $(-1,1)$ into sub-intervals such that the transformation $y = x^2$ is one-to-one in each sub-interval. Thus

$\underline{-1 \leqslant x \leqslant 0}$: $\quad y = x^2 \Rightarrow x = -\sqrt{y}, \quad \dfrac{dy}{dx} = 2x = -2\sqrt{y}$:

$$f_Y^-(y) = f_X^-(-\sqrt{y})\left|\frac{dx}{dy}\right| = \tfrac{1}{2}(1-\sqrt{y}).\frac{1}{2\sqrt{y}}, \quad -1 \leqslant -\sqrt{y} \leqslant 0, \text{ i.e. } 0 \leqslant \sqrt{y} \leqslant 1.$$

$\underline{0 \leqslant x \leqslant 1}$: $\quad y = x^2 \Rightarrow x = +\sqrt{y}, \quad \dfrac{dy}{dx} = 2x = 2\sqrt{y}$:

$$f_Y^+(y) = f_X^+(+\sqrt{y})\left|\frac{dx}{dy}\right| = \tfrac{1}{2}(1+\sqrt{y}).\frac{1}{2\sqrt{y}}, \quad 0 \leqslant \sqrt{y} \leqslant 1.$$

/continued overleaf

Then

$$\begin{aligned} f_Y(y) &= f_Y^+(y) + f_Y^-(y) \\ &= \begin{cases} \dfrac{1}{2\sqrt{y}}, & 0 \leqslant y \leqslant 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(c)

$$f_X(x) = \begin{cases} \frac{1}{2}, & -\frac{1}{2} \leqslant x \leqslant \frac{3}{2} \\ 0, & \text{otherwise.} \end{cases}$$

The transformation $y = x^2$ is *not* one-to-one when $-\frac{1}{2} \leqslant x \leqslant \frac{3}{2}$. So

EITHER

$$\mathrm{P}(Y \leqslant y) = \mathrm{P}(X^2 \leqslant y) = \mathrm{P}(-\sqrt{y} \leqslant X \leqslant +\sqrt{y}) = \int_{-\sqrt{y}}^{+\sqrt{y}} f_X(x)dx.$$

Now

$$\begin{aligned} \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx &= \int_{-\sqrt{y}}^{\sqrt{y}} \tfrac{1}{2}dx \quad \underline{\text{provided}} \qquad -\tfrac{1}{2} \leqslant -\sqrt{y} \leqslant \sqrt{y} \leqslant \tfrac{1}{2} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{i.e. } 0 \leqslant y \leqslant \tfrac{1}{4} \\ &= \left[\tfrac{1}{2}x\right]_{-\sqrt{y}}^{\sqrt{y}} = \sqrt{y}, \qquad 0 \leqslant y \leqslant \tfrac{1}{4} \end{aligned}$$

while

$$\begin{aligned} \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx &= \int_{-\frac{1}{2}}^{\sqrt{y}} \tfrac{1}{2}dx \quad \underline{\text{provided}} \qquad -\tfrac{3}{2} \leqslant -\sqrt{y} \leqslant -\tfrac{1}{2} \leqslant \tfrac{1}{2} < \sqrt{y} \leqslant \tfrac{3}{2} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{i.e. } \tfrac{1}{4} < y \leqslant \tfrac{9}{4} \\ &= \left[\tfrac{1}{2}x\right]_{-\frac{1}{2}}^{\sqrt{y}} = \tfrac{1}{2}\sqrt{y} + \tfrac{1}{4}, \qquad \tfrac{1}{4} < y \leqslant \tfrac{9}{4} \end{aligned}$$

so that

$$F_Y(y) = \begin{cases} \sqrt{y}, & 0 \leqslant y \leqslant \frac{1}{4} \\ \frac{1}{2}\sqrt{y} + \frac{1}{4}. & \frac{1}{4} < y \leqslant \frac{9}{4}. \end{cases}$$

Hence

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}, & 0 \leqslant y \leqslant \frac{1}{4} \\ \dfrac{1}{4\sqrt{y}}, & \frac{1}{4} < y \leqslant \frac{9}{4}. \end{cases}$$

OR

$\underline{-\frac{1}{2} \leqslant x \leqslant 0}:$ $\quad y = x^2 \Rightarrow x = -\sqrt{y}, \quad \dfrac{dy}{dx} = 2x = -2\sqrt{y}$:

$$f_Y^-(y) = f_X^-(-\sqrt{y}) \left|\frac{dx}{dy}\right| = \frac{1}{2}.\frac{1}{2\sqrt{y}}, \quad -\tfrac{1}{2} \leqslant -\sqrt{y} \leqslant 0 \ \text{ or } \ 0 \leqslant y \leqslant \tfrac{1}{4}.$$

$\underline{0 \leqslant x \leqslant \frac{3}{2}}:$ $\quad y = x^2 \Rightarrow x = +\sqrt{y}, \quad \dfrac{dy}{dx} = 2x = 2\sqrt{y}$:

$$f_Y^+(y) = f_X^+(+\sqrt{y}) \left|\frac{dx}{dy}\right| = \frac{1}{2}.\frac{1}{2\sqrt{y}}, \quad 0 \leqslant \sqrt{y} \leqslant \tfrac{3}{2} \ \text{ or } \ 0 \leqslant y \leqslant \tfrac{9}{4}.$$

Then

$$\begin{aligned} f_Y(y) &= f_Y^+(y) + f_Y^-(y) \\ &= \begin{cases} \dfrac{1}{2\sqrt{y}}, & 0 \leqslant y \leqslant \frac{1}{4} \\ \dfrac{1}{4\sqrt{y}}, & \frac{1}{4} < y \leqslant \frac{9}{4}. \end{cases} \end{aligned}$$

/continued overleaf

(ii) $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \qquad -\infty < z < \infty.$

We note that the transformation $v = z^2$ is *not* one-to-one when $-\infty < z < \infty$; $0 \leqslant v < \infty$. Then

EITHER

$$\begin{aligned} F_V(v) &= \mathrm{P}(V \leqslant v) = \mathrm{P}(Z^2 \leqslant v) \\ &= \mathrm{P}(-\sqrt{v} \leqslant Z \leqslant +\sqrt{v}) = F_Z(+\sqrt{v}) - F_Z(-\sqrt{v}). \end{aligned}$$

So

$$\begin{aligned} f_V(v) &= \frac{dF_V(v)}{dv} = \frac{d}{dv}\left\{F_Z(\sqrt{v}) - F_Z(-\sqrt{v})\right\} \\ &= f_Z(\sqrt{v}).\frac{1}{2\sqrt{v}} - f_Z(-\sqrt{v})\left(-\frac{1}{2\sqrt{v}}\right) \\ &= \frac{1}{2\sqrt{v}}\left\{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v}\right\} \\ &= \frac{1}{2^{\frac{1}{2}}\sqrt{\pi}} v^{-\frac{1}{2}} e^{-\frac{1}{2}v} \\ &= \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})} v^{\frac{1}{2}-1} e^{-\frac{1}{2}v}, \qquad 0 \leqslant v < \infty \end{aligned}$$

which is the p.d.f. of the $\chi^2(1)$ distribution,

i.e. $V \sim \chi^2(1).$

OR

Consider the one-to-one transformations for $-\infty < z < 0$ and $0 \leqslant z < \infty$.

$\underline{-\infty < z < 0}$: $\quad v = z^2 \Rightarrow z = -\sqrt{v}, \quad \frac{dv}{dz} = 2z = -2\sqrt{v}$:

$$\therefore\ f_V^-(v) = f_Z^-(-\sqrt{v})\left|\frac{dz}{dv}\right| = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v}\frac{1}{2\sqrt{v}}, \quad 0 \leqslant v < \infty.$$

$\underline{0 \leqslant z < \infty}$: $\quad v = z^2 \Rightarrow z = +\sqrt{v}, \quad \frac{dv}{dz} = 2z = 2\sqrt{v}$:

$$\therefore\ f_V^+(v) = f_Z^+(+\sqrt{v})\left|\frac{dz}{dv}\right| = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v}\frac{1}{2\sqrt{v}}, \quad 0 \leqslant v < \infty.$$

So $\quad f_V(v) = f_V^+(v) + f_V^-(v) = \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}v}\frac{1}{2\sqrt{v}} \quad \longrightarrow$ above result.

(iii) $f_X(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$

The transformation $y = x^\alpha$ is one-to-one when $0 \leqslant x \leqslant 1$, and

$$\frac{dy}{dx} = \alpha x^{\alpha-1} = \alpha y^{(\alpha-1)/\alpha}.$$

So

$$\begin{aligned} f_Y(y) &= f_X(y^{1/\alpha})\left|\frac{dx}{dy}\right| = \begin{cases} 1.\left|\frac{1}{\alpha y^{(1-\alpha)/\alpha}}\right|, & 0 \leqslant y^{1/\alpha} \leqslant 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{\alpha} y^{(1/\alpha)-1}, & 0 \leqslant y \leqslant 1;\ \ \alpha > 0 \\ -\frac{1}{\alpha} y^{(1/\alpha)-1)}, & 1 \leqslant y < \infty;\ \ \alpha < 0. \end{cases} \end{aligned}$$

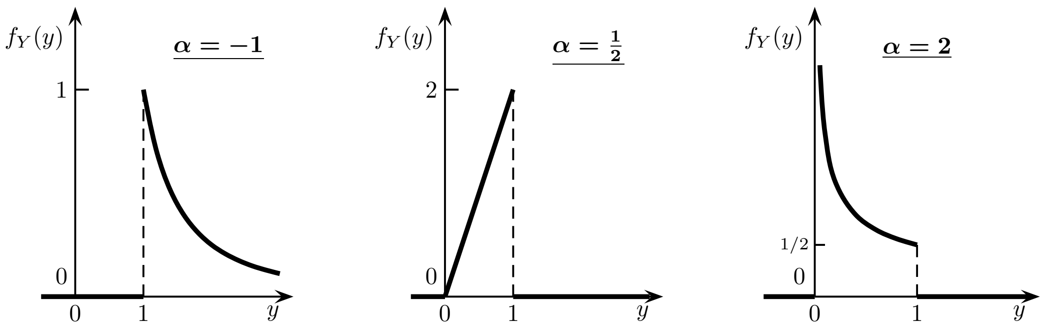
For $\underline{\alpha > 0}$:

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1, & y > 1 \\ \displaystyle\int_0^y \frac{1}{\alpha} t^{(1/\alpha)-1} dt = y^{1/\alpha}, & 0 \leqslant y \leqslant 1. \end{cases}$$

For $\underline{\alpha < 0}$:

$$F_Y(y) = \begin{cases} 0, & y < 1 \\ \displaystyle\int_1^y (-\frac{1}{\alpha}) t^{(1/\alpha)-1} dt = y^{1/\alpha}, & y \geqslant 1. \end{cases}$$

For the three cases asked for, we then have:



$f_Y(y) = \dfrac{1}{y^2},\ 1 \le y < \infty$ $\qquad$ $f_Y(y) = 2y,\ 0 \le y \le 1$ $\qquad$ $f_Y(y) = \frac{1}{2} y^{-1/2},\ 0 \le y \le 1$

4. (i) <u>Normal distribution N(μ, σ^2)</u>

(a)

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x) dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\tfrac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx \qquad \text{[set } z = \tfrac{x-\mu}{\sigma}\text{]} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \int_{\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} (-dy) + \qquad \text{ditto} \qquad \text{[setting } z = -y\text{]} \\
&= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \qquad \text{[set } t = \tfrac{1}{2}y^2, y = \sqrt{2t}, \\
&\qquad\qquad dt = y\,dy = \sqrt{2t}\,dy \quad \text{]} \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} \frac{1}{\sqrt{2t}} dt \\
&= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{1}{2}-1} e^{-t} dt = \frac{1}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}) = 1. \qquad \text{[using } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\text{]}
\end{aligned}$$

/continued overleaf

(b) The transformation $w = a + bx$ is one-to-one and differentiable;

$$\frac{dw}{dx} = b, \quad x = \frac{w-a}{b}.$$

Then

$$\begin{aligned} f_W(w) &= f_X\left(\frac{w-a}{b}\right)\left|\frac{dx}{dw}\right| \\ &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\tfrac{1}{2}\left(\frac{\frac{w-a}{b}-\mu}{\sigma}\right)^2\right\}\left|\frac{1}{b}\right|, \quad -\infty < \tfrac{w-a}{b} < \infty \\ &= \frac{1}{\sqrt{2\pi}\sigma|b|}\exp\left\{-\tfrac{1}{2}\left(\frac{w-a-\mu b}{b\sigma}\right)^2\right\}, \quad -\infty < w < \infty \end{aligned}$$

i.e. $W \sim \mathrm{N}(a+b\mu, b^2\sigma^2)$, and $\mathrm{SD}(W) = \sqrt{b^2\sigma^2} = |b|\sigma$.

(ii) Negative exponential distribution (λ)

(a)

$$\begin{aligned} \mathrm{E}(X) &= \int_0^\infty x.\lambda e^{-\lambda x}dx \qquad [\text{set } t = \lambda x,\ dt = \lambda dx] \\ &= \int_0^\infty \left(\frac{t}{\lambda}\right)\lambda e^{-t}.\frac{1}{\lambda}dt = \frac{1}{\lambda}\int_0^\infty t^{2-1}e^{-t}dt \\ &= \frac{1}{\lambda}\Gamma(2) = \frac{1}{\lambda}1! = \frac{1}{\lambda}. \end{aligned}$$

Similarly $\mathrm{E}(X^2) = \frac{1}{\lambda^2}\Gamma(3) = \frac{1}{\lambda^2}2! = \frac{2}{\lambda^2}.$

So $\mathrm{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$

(b)

$$\begin{aligned} \mathrm{P}(X > s+t | X > s) &= \frac{\mathrm{P}(X > s+t)}{\mathrm{P}(X > s)} \qquad \text{when } t > 0 \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \qquad [\text{since } \mathrm{P}(X > y) = 1 - \mathrm{P}(X \leqslant y) \\ &\qquad\qquad = 1 - \{1 - e^{-\lambda y}\} = e^{-\lambda y}] \\ &= e^{-\lambda t} = \mathrm{P}(X > t). \end{aligned}$$

(iii) Gamma distribution (α, λ)

(a)

$$\begin{aligned} \mathrm{E}(X) &= \int_0^\infty x\frac{\lambda^\alpha x^{\alpha-1}\exp(-\lambda x)}{\Gamma(\alpha)}dx \qquad [\text{set } t = \lambda x, dt = \lambda dx] \\ &= \int_0^\infty \left(\frac{t}{\lambda}\right)\frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}dt \\ &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^\infty t^{(\alpha+1)-1}e^{-t}dt \\ &= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda} \qquad [\text{since } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)]. \end{aligned}$$

Similarly $\mathrm{E}(X^2) = \frac{\Gamma(\alpha+2)}{\lambda^2\Gamma(\alpha)} = \frac{(\alpha+1)\alpha}{\lambda^2}.$

So $\mathrm{Var}(X) = \frac{(\alpha+1)\alpha}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}.$

/continued overleaf

(b) $V \sim \chi^2(r) = $ Gamma $(\alpha = \frac{r}{2}, \lambda = \frac{1}{2})$.

So $\quad f_V(v) = \dfrac{v^{\frac{r}{2}-1}\exp(-\frac{v}{2})}{2^{\frac{r}{2}}\Gamma(\frac{r}{2})}, \quad v \geqslant 0.$

$\mathrm{E}(V) = \dfrac{\frac{r}{2}}{\frac{1}{2}} = r; \quad \mathrm{Var}(V) = \dfrac{\frac{r}{2}}{(\frac{1}{2})^2} = 2r.$

(iv) Beta distribution (a, b)

(a)

$$\begin{aligned} \mathrm{E}(X) &= \int_0^1 x.\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}dx \\ &= \frac{1}{B(a,b)}\int_0^1 x^a(1-x)^{b-1}dx \\ &= \frac{B(a+1,b)}{B(a,b)} = \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}.\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = \frac{a}{a+b}. \end{aligned}$$

Similarly

$$\begin{aligned} \mathrm{E}(X^2) &= \frac{B(a+2,b)}{B(a,b)} \\ &= \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)}.\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = \frac{(a+1)a}{(a+b+1)(a+b)}. \end{aligned}$$

So

$$\begin{aligned} \mathrm{Var}(X) &= \frac{(a+1)a}{(a+b+1)(a+b)} - \left(\frac{a}{a+b}\right)^2 \\ &= \frac{(a+1)a(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)} \\ &= \frac{ab}{(a+b)^2(a+b+1)}. \end{aligned}$$

(b) The transformation $y = 1 - x$ is one-to-one and differentiable;

$$x = 1 - y; \quad \frac{dy}{dx} = -1$$

Then

$$\begin{aligned} f_Y(y) &= f_X(1-y)|-1|, \qquad 0 \leqslant y \leqslant 1 \\ &= \frac{1}{B(a,b)}(1-y)^{a-1}y^{b-1}, \qquad 0 \leqslant y \leqslant 1 \end{aligned}$$

i.e. $\qquad Y \sim$ beta (b, a).

(c) Transforming $0 \leqslant x \leqslant 1$ to $A \leqslant w \leqslant B$ involves a *location* change $0 \to A$ and a *scale* change of $1 - 0 \to B - A$, i.e. a linear transformation

$$w = A + (B - A)x.$$

This is a one-to-one differentiable transformation;

$$x = (w - A)/(B - A); \quad \frac{dw}{dx} = B - A.$$

So

$$\begin{aligned} f_W(w) &= f_X\left\{\frac{w-A}{B-A}\right\}\left|\frac{dx}{dw}\right| \\ &= \frac{1}{B(a,b)}\left(\frac{w-A}{B-A}\right)^{a-1}\left(1 - \frac{w-A}{B-A}\right)^{b-1}.\frac{1}{B-A}, \quad A \leqslant w \leqslant B \\ &= \frac{1}{B(a,b)}.\frac{(w-A)^{a-1}(B-w)^{b-1}}{(B-A)^{a+b-1}}, \quad A \leqslant w \leqslant B. \end{aligned}$$

(v) Weibull distribution (b, c)

(a) $F(x) = 1 - \exp\left\{-\left(\frac{x}{b}\right)^c\right\}, \quad x \geqslant 0.$

So $\quad f_X(x) = \frac{dF}{dx} = \frac{cx^{c-1}}{b^c} \exp\left\{-\left(\frac{x}{b}\right)^c\right\}, \quad x \geqslant 0.$

$$\begin{aligned} \mathrm{E}(X) &= \int_0^\infty x \frac{cx^{c-1}}{b^c} \exp\left\{-\left(\frac{x}{b}\right)^c\right\} dx \quad [\text{set } t = (\tfrac{x}{b})^c, dt = \tfrac{cx^{c-1}}{b^c} dx, x = bt^{1/c}] \\ &= \int_0^\infty bt^{1/c} e^{-t} dt = b\Gamma\left(1 + 1/c\right). \end{aligned}$$

(b) *Survival function* $\quad \overline{F}(x) = 1 - F(x) = \exp\left\{-\left(\frac{x}{b}\right)^c\right\}, \quad x \geqslant 0.$

Hazard function

$$H(x) = -\log_e(1 - F(x)) = -\log_e\left[\exp\left\{-\left(\frac{x}{b}\right)^c\right\}\right] = \left(\frac{x}{b}\right)^c, \quad x \geqslant 0.$$

Hazard rate function $\quad r(x) = \frac{f(x)}{\overline{F}(x)} = \frac{cx^{c-1}}{b^c}, \quad x \geqslant 0.$

For $c < 1$, $\quad r(x)$ is a decreasing frunction of x;
For $c = 1$, $\quad r(x) = 1/b$, i.e. a constant;
For $c > 1$, $\quad r(x)$ is an increasing function of x.