SOR201 Solutions to Examples 7

1. (i) The joint p.d.f. of (X, Y, Z) is

$$f(x,y,z) = \begin{cases} Kxyz^2, & 0 \leqslant x, y \leqslant 1,\ 0 \leqslant z \leqslant 3 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Integrating over all three variables:

$$\begin{aligned} 1 &= \int_0^1 \int_0^1 \int_0^3 Kxyz^2 dxdydz \qquad \text{[order of integration is not important]} \\ &= \int_0^1 \int_0^1 Kxy \left[\tfrac{1}{3}z^3\right]_{z=0}^{z=3} dxdy \\ &= \int_0^1 \int_0^1 9Kxydxdy = \int_0^1 9Kx \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1} dx \\ &= \int_0^1 \tfrac{9}{2}Kxdx = \tfrac{9}{2}K \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{9}{4}K. \end{aligned}$$

So $K = \frac{4}{9}$.

(b) The marginal p.d.f. of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y,z)dxdz \\ &= \int_0^1 \int_0^3 \tfrac{4}{9}xyz^2 dxdz = \int_0^1 \tfrac{4}{9}xy \left[\tfrac{1}{3}z^3\right]_{z=0}^{z=3} \\ &= \int_0^1 4xydx, \qquad 0 \leqslant y \leqslant 1 \\ &= y\left[2x^2\right]_{x=0}^{x=1} = \begin{cases} 2y, & 0 \leqslant y \leqslant 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_0^1 2y^2dy = \left[\tfrac{2}{3}y^3\right]_0^1 = \tfrac{2}{3}.$$

(c) The joint p.d.f. of (X, Z) is

$$\begin{aligned} f_{X,Z}(x,z) &= \int_{-\infty}^{\infty} f(x,y,z)dy \\ &= \int_0^1 \tfrac{4}{9}xyz^2dy, \qquad 0 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 3 \\ &= \tfrac{4}{9}xz^2 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1} \\ &= \begin{cases} \tfrac{2}{9}xz^2, & 0 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 3 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(d) The conditional p.d.f. of Y given $X = \frac{1}{2}, Z = 1$ is

$$\begin{aligned} f(y|X=\tfrac{1}{2}, Z=1) &= \frac{f(\frac{1}{2},y,1)}{f_{X,Z}(\frac{1}{2},1)} \\ &= \frac{\frac{4}{9}\times\frac{1}{2}y\times 1^2}{\frac{2}{9}\times\frac{1}{2}\times 1^2} = \begin{cases} 2y, & 0 \leqslant y \leqslant 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{E}(Y|X=\tfrac{1}{2}, Z=1) &= \int_{-\infty}^{\infty} yf(y|X=\tfrac{1}{2}, Z=1)dy \\ &= \int_0^1 y.2ydy = \left[\tfrac{2}{3}y^3\right]_0^1 = \tfrac{2}{3}. \end{aligned}$$

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OR

Since $f(x,y,z) =$ (function of x)$\times$(function of y)$\times$(function of z) for all (x,y,z), it follows that X,Y and Z are independent random variables.

So $\qquad f_X(x) = ax, \qquad 0 \leqslant x \leqslant 1,$

where the constant a is such that $\int_{-\infty}^{\infty} f_X(x)dx = \int_0^1 axdx = 1$, i.e. $a = 2$.

Similarly,

$$\begin{array}{rclccc} f_Y(y) &=& by, & 0 \leqslant y \leqslant 1 & \Rightarrow & b = 2 \\ f_Z(z) &=& cz^2, & 0 \leqslant z \leqslant 3 & \Rightarrow & c = \frac{1}{9}. \end{array}$$

Hence

(a) $K = abc = \frac{4}{9}$

(b) $f_Y(y) = 2y, \quad 0 \leqslant y \leqslant 1: \qquad \mathrm{E}(Y) = \frac{2}{3}$

(c) $f_{X,Z}(x,z) = f_X(x).f_Z(z)$ since X,Z are independent, i.e.

$$f_{X,Z}(x,z) = \begin{cases} \frac{2}{9}xz^2, & 0 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 3 \\ 0, & \text{otherwise.} \end{cases}$$

(d) Since Y is independent of X,Z

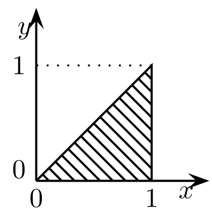
$$f(y|X = \tfrac{1}{2}, Z = 1) = f_Y(y) = 2y, \qquad 0 \leqslant y \leqslant 1.$$

and

$$\mathrm{E}(Y|X = \tfrac{1}{2}, Z = 1) = \mathrm{E}(Y) = \tfrac{2}{3}.$$

(ii) $f(x,y) = K(1-x)^\alpha y^\beta, \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x; \quad \alpha,\beta > -1.$

(a) $f(x,y) > 0$ in the shaded area $0 \leqslant y \leqslant x \leqslant 1$ (except along the lines $x = 1$ and $y = 0$)



(b) The marginal distribution of X has p.d.f.

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x K(1-x)^\alpha y^\beta dy, \qquad 0 \leqslant x \leqslant 1.$$

Consider x to be fixed, $0 \leqslant x \leqslant 1$; then $0 \leqslant y \leqslant x$.
So

$$\begin{aligned} g(x) &= K(1-x)^\alpha \left[\frac{1}{\beta+1} y^{\beta+1}\right]_{y=0}^{y=x} \\ &= \frac{K}{\beta+1}(1-x)^\alpha x^{\beta+1}, \qquad 0 \leqslant x \leqslant 1. \end{aligned}$$

Recognising this as the p.d.f. of a beta distribution, it follows that

$$g(x) = \frac{1}{B(\beta+2,\alpha+1)} x^{(\beta+2)-1}(1-x)^{(\alpha+1)-1}, \quad 0 \leqslant x \leqslant 1,$$

where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad (a,b > 0)$$

(see eqns. (5.27), (5.28) of lecture notes).

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Hence $\dfrac{K}{\beta+1} = \dfrac{1}{B(\beta+2,\alpha+1)}$ or $K = \dfrac{(\beta+1)}{B(\beta+2,\alpha+1)}$.

Then

$$\begin{aligned} f(y|x) = \frac{f(x,y)}{g(x)} &= \frac{K(1-x)^\alpha y^\beta}{\frac{K}{\beta+1}(1-x)^\alpha x^{\beta+1}} \\ &= \frac{(\beta+1)y^\beta}{x^{\beta+1}}, \qquad 0 \leqslant y \leqslant x \leqslant 1 \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}(Y|X) &= \int_{-\infty}^{\infty} y f(y|x)dy = \int_0^x y.\frac{(\beta+1)y^\beta}{x^{\beta+1}}dy \\ &= \frac{(\beta+1)}{x^{\beta+1}}\left[\frac{y^{\beta+2}}{\beta+2}\right]_{y=0}^{y=x} = \frac{(\beta+1)}{(\beta+2)}x, \qquad 0 \leqslant x \leqslant 1. \end{aligned}$$

(c) The marginal distribution of Y has p.d.f.

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 K(1-x)^\alpha y^\beta dx, \qquad 0 \leqslant y \leqslant 1.$$

Consider y to be fixed, $0 \leqslant y \leqslant 1$; then $y \leqslant x \leqslant 1$.
So

$$\begin{aligned} h(y) &= Ky^\beta \left[\left(-\frac{1}{\alpha+1}\right)(1-x)^{\alpha+1}\right]_{x=y}^{x=1} \\ &= \frac{K}{\alpha+1}(1-y)^{\alpha+1}y^\beta, \qquad 0 \leqslant y \leqslant 1 \\ &= \frac{(\beta+1)}{(\alpha+1)B(\beta+2,\alpha+1)}y^\beta(1-y)^{\alpha+1}, \quad 0 \leqslant y \leqslant 1. \end{aligned}$$

Then

$$\begin{aligned} f(x|y) = \frac{f(x,y)}{h(y)} &= \frac{K(1-x)^\alpha y^\beta}{\frac{K}{\alpha+1}(1-y)^{\alpha+1}y^\beta} \\ &= \frac{(\alpha+1)(1-x)^\alpha}{(1-y)^{\alpha+1}}, \quad 0 \leqslant y \leqslant x \leqslant 1. \end{aligned}$$

Also

$$\begin{aligned} \mathrm{E}(X|y) &= \int_{-\infty}^{\infty} x f(x|y)dx \\ &= \int_y^1 x.\frac{(\alpha+1)(1-x)^\alpha}{(1-y)^{\alpha+1}}dx \\ &= \frac{(\alpha+1)}{(1-y)^{\alpha+1}}\int_y^1 x(1-x)^\alpha dx. \end{aligned}$$

Integrating by parts:

$$\begin{aligned} \int_y^1 x(1-x)^\alpha dx &= \left[-\frac{(1-x)^{\alpha+1}}{(\alpha+1)}.x\right]_y^1 + \int_y^1 \frac{(1-x)^{\alpha+1}}{(\alpha+1)}dx \\ &= \frac{(1-y)^{\alpha+1}}{(\alpha+1)}\left[y + \frac{(1-y)}{(\alpha+2)}\right]. \end{aligned}$$

So $\mathrm{E}(X|y) = y + \dfrac{(1-y)}{(\alpha+2)}$.

2. (a) The joint p.d.f. of (X, Y) is

$$\begin{aligned} f_{X,Y} &= f_X(x).f_y(y) \qquad \text{[since } X, Y \text{ are independent]} \\ &= \begin{cases} \dfrac{\lambda^{\alpha+\beta} x^{\alpha-1} y^{\beta-1} e^{-\lambda(x+y)}}{\Gamma(\alpha)\Gamma(\beta)}, & 0 \leqslant x, y < \infty \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $U = X + Y, \quad V = X/Y.$

The transformation $u = x + y, \quad v = x/y, \quad 0 \leqslant x, y < \infty$

is one-to-one with inverse

$$x = \frac{uv}{1+v}, \quad y = \frac{u}{1+v}.$$

The Jacobian of the (original) transformation is

$$J(u, v; x, y) = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = -\frac{(x+y)}{y^2} = -\frac{(1+v)^2}{u}.$$

So the joint p.d.f. of (U, V) is

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}\left(\frac{uv}{1+v}, \frac{u}{1+v}\right) |J(x, y; u, v)| \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{uv}{1+v}\right)^{\alpha-1} \left(\frac{u}{1+v}\right)^{\beta-1} e^{-\lambda u} \left| -\frac{u}{(1+v)^2} \right|, \\ & \qquad \text{when } 0 \leqslant \frac{uv}{1+v}, \frac{u}{1+v} < \infty \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha+\beta-1} e^{-\lambda u} \frac{v^{\alpha-1}}{(1+v)^{\alpha+\beta}}, \quad 0 \leqslant u, v < \infty \\ &= \text{(function of } u) \times \text{(function of } v), \quad \text{for all } (u, v). \end{aligned}$$

So U and V are independent random variables, i.e. $X+Y$ and X/Y are independent random variables.

The p.d.f. of $U = X + Y$ is proportional to $u^{\alpha+\beta-1} e^{-\lambda u}, \ 0 \leqslant u < \infty.$

Hence $U \sim \text{gamma}(\alpha + \beta, \lambda)$ and so

$$f_U(u) = \frac{\lambda^{\alpha+\beta} u^{\alpha+\beta-1} e^{-\lambda u}}{\Gamma(\alpha+\beta)}, \quad 0 \leqslant u < \infty.$$

The p.d.f. of $V = X/Y$ is then

$$f_V(v) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{v^{\alpha-1}}{(1+v)^{\alpha+\beta}}, \quad 0 \leqslant v < \infty.$$

Now reverse the roles of X and Y in the above discussion. Then we have that

$Y + X$ and Y/X are independent random variables;

$[Y + X \sim \text{gamma}(\beta + \alpha, \lambda);]$

Y/X has p.d.f. $\dfrac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)} \dfrac{w^{\beta-1}}{(1+w)^{\beta+\alpha}}, \quad 0 \leqslant w < \infty.$

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(b) From the end of part(a), Y/X and $Y+X$ are independent random variables.

So $\dfrac{1}{1+Y/X} = \dfrac{X}{X+Y}$ and $X+Y$ are independent random variables.

Also, $W = Y/X$ has p.d.f. $f_W(w) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \dfrac{w^{\beta-1}}{(1+w)^{\alpha+\beta}}, \quad 0 \leqslant w < \infty.$

The transformation $t = \dfrac{1}{1+w}, \; 0 \leqslant w < \infty$ is one-to-one and differentiable:

$\dfrac{dt}{dw} = -\dfrac{1}{(1+w)^2} = -t^2;$ the inverse is $w = \dfrac{1-t}{t}.$

Hence

$$
\begin{aligned}
f_T(t) &= f_W\left(\frac{1-t}{t}\right)\left|\frac{dw}{dt}\right| \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(1-t)^{\beta-1}}{t^{\beta-1}} t^{\alpha+\beta}\left|-\frac{1}{t^2}\right|, \quad 0 \leqslant \frac{1-t}{t} < \infty \\
&= \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}, \quad 0 \leqslant t \leqslant 1
\end{aligned}
$$

i.e. $\dfrac{X}{X+Y} \sim \text{beta}(\alpha,\beta).$

(c) Negative exponential $(\lambda) \equiv$ gamma $(1,\lambda)$.

So $\{X, Y$ are independent negative exponential (λ) random variables$\}$

$\Rightarrow X+Y, \; X/Y$ are independent random variables, where

$$X+Y \sim \text{gamma}(2,\lambda);$$

$$\text{p.d.f. of } V = X/Y \text{ is } \frac{1}{(1+v)^2}, \qquad 0 \leqslant v < \infty;$$

$$\frac{X}{X+Y} \sim \text{beta}(1,1) = \text{uniform}[0,1].$$

(d) $\chi^2(r) \equiv \text{gamma}(\tfrac{1}{2}r, \tfrac{1}{2}), \quad r$ a positive integer.

So $\{X, Y$ are independent $\chi^2(r_1), \chi^2(r_2)$ random variables$\}$

$\Rightarrow X+Y, X/Y$ are independent random variables, where

$$X+Y \sim \text{gamma}(r_1+r_2, \tfrac{1}{2}) = \chi^2(r_1+r_2);$$

$$\text{p.d.f. of } V = X/Y \text{ is } \frac{\Gamma(\frac{1}{2}r_1+\frac{1}{2}r_2)}{\Gamma(\frac{1}{2}r_1)\Gamma(\frac{1}{2}r_2)} \frac{v^{\frac{1}{2}r_1-1}}{(1+v)^{\frac{1}{2}r_1+\frac{1}{2}r_2}}, \quad 0 \leqslant v < \infty;$$

$$\frac{X}{X+Y} \sim \text{beta}(\tfrac{1}{2}r_1, \tfrac{1}{2}r_2).$$

3. (i) The joint p.d.f. of (X, Y) is

$$f_{X,Y}(x,y) = f_X(x).f_Y(y), \qquad -\infty < x, y < \infty.$$

(a) The transformation $u = xy,\ v = y$ is one-to-one and has inverse $x = u/v,\ y = v$. The Jacobian of the (original) transformation is

$$J(u,v;x,y) = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y = v.$$

So the joint p.d.f. of (U, V) is

$$f_{U,V}(u,v) = f_X\left(\frac{u}{v}\right) f_Y(v) \left|\frac{1}{v}\right|, \qquad -\infty < u, v < \infty$$

and the p.d.f. of $U = XY$ is then

$$\int_{-\infty}^{\infty} f_X\left(\frac{u}{v}\right) f_Y(v) \left|\frac{1}{v}\right| dv, \qquad -\infty < u < \infty.$$

<u>Alternatively</u>: the transformation $u = xy, v = x$ is one-to-one and has inverse $x = v,\ y = u/v$. The Jacobian of the (original) transformation is

$$J(u,v;x,y) = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -v.$$

So the joint p.d.f. of (U, V) is

$$f_{U,V}(u,v) = f_X(v) f_Y\left(\frac{u}{v}\right) \left|-\frac{1}{v}\right|, \qquad -\infty < u, v < \infty$$

and the p.d.f. of $U = XY$ is then

$$\int_{-\infty}^{\infty} f_X(v) f_Y\left(\frac{u}{v}\right) \left|\frac{1}{v}\right| dv, \qquad -\infty < u < \infty.$$

(b) The transformation $u = x/y,\ v = y$ is one-to-one and has inverse $x = uv,\ y = v$. The Jacobian of the inverse transformation is

$$J(x,y;u,v) = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$

So the joint p.d.f. of (U, V) is

$$f_{U,V}(u,v) = f_X(uv) f_Y(v) |v|, \qquad -\infty < u, v < \infty.$$

and the p.d.f. of $U = X/Y$ is then

$$\int_{-\infty}^{\infty} f_X(uv) f_Y(v) |v| dv, \qquad -\infty < u < \infty.$$

/continued overleaf

(c) The transformation $u = x + y,\ v = x$ is one-to-one and has inverse $x = v,\ y = u - v$. The Jacobian of the (original) transformation is

$$J(u, v; x, y) = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

So the joint p.d.f. of (U, V) is

$$f_{U,V}(u, v) = f_X(v) f_Y V(u - v)| - 1|, \qquad -\infty < u, v < \infty$$

and the p.d.f. of $U = X + Y$ is

$$\int_{-\infty}^{\infty} f_X(v) f_Y(u - v) dv, \qquad -\infty < u < \infty.$$

Alternatively: the transformation $u = x + y,\ v = y$ is one-to-one with inverse $x = u - v,\ y = v$. The Jacobian of the (original) transformation is

$$J(u, v; x, y) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$$

So the joint p.d.f. of (U, V) is

$$f_{U,V}(u, v) = f_X(u - v) f_Y(v)|1|, \qquad -\infty < u, v < \infty$$

and the p.d.f. of $U = X + Y$ is then

$$\int_{-\infty}^{\infty} f_X(u - v) f_Y(v) dv, \qquad -\infty < u < \infty.$$

(ii) $f_X(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise;} \end{cases} \qquad f_Y(y) = \begin{cases} 1, & 0 \leqslant y \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$

(a) The p.d.f. of $U = XY$ is $\int_{-\infty}^{\infty} f_X\left(\frac{u}{v}\right) f_Y(v) \left|\frac{1}{v}\right| dv$, where

$$f_X\left(\frac{u}{v}\right) f_Y(v) = \begin{cases} 1, & \text{when } 0 \leqslant u/v \leqslant 1,\ 0 \leqslant v \leqslant 1 \quad \text{i.e.} \quad 0 \leqslant u \leqslant v \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

So

$$f_U(u) = \int_u^1 \frac{1}{v} dv = [\log_e v]_u^1 = -\log_e u, \quad 0 \leqslant u \leqslant 1.$$

/continued overleaf

(b) The p.d.f. of $U = X/Y$ is $\displaystyle\int_{-\infty}^{\infty} f_X(uv)f_Y(v)|v|dv$,

where

$$f_X(uv)f_Y(v) = \begin{cases} 1, & \text{when } 0 \leqslant uv \leqslant 1,\ 0 \leqslant v \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$0 \leqslant uv \leqslant 1,\ 0 \leqslant v \leqslant 1 \Rightarrow \begin{cases} 0 \leqslant v \leqslant 1 \leqslant 1/u & \text{when } 0 \leqslant u \leqslant 1 \\ 0 \leqslant v \leqslant 1/u \leqslant 1 & \text{when } 1 \leqslant u < \infty. \end{cases}$$

So

$$f_U(u) = \begin{cases} \displaystyle\int_0^1 vdv = \left[\tfrac{1}{2}v^2\right]_0^1 = \tfrac{1}{2}, & \text{when } 0 \leqslant u \leqslant 1 \\ \displaystyle\int_0^{1/u} vdv = \frac{1}{2u^2}, & \text{when } 1 \leqslant u < \infty. \end{cases}$$

(c) The p.d.f. of $U = X + Y$ is $\displaystyle\int_{-\infty}^{\infty} f_X(u-v)f_Y(v)dv$,

where

$$f_X(u-v)f_Y(v) = \begin{cases} 1, & \text{when } 0 \leqslant u-v \leqslant 1,\ 0 \leqslant v \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$0 \leqslant u-v \leqslant 1,\ 0 \leqslant v \leqslant 1 \Rightarrow \begin{cases} 0 \leqslant v \leqslant u \leqslant 1, & \text{when } 0 \leqslant u \leqslant 1 \\ u-1 \leqslant v \leqslant 1, & \text{when } 1 \leqslant u \leqslant 2. \end{cases}$$

So

$$f_U(u) = \begin{cases} \displaystyle\int_0^u 1dv = u, & \text{when } 0 \leqslant u \leqslant 1 \\ \displaystyle\int_{u-1}^1 1dv = [v]_{u-1}^1 = 2-u, & \text{when } 1 \leqslant u \leqslant 2. \end{cases}$$

4. (i) The joint p.d.f. of (X_1, X_2, X_3) is

$$\begin{aligned} f_{X_1,X_2,X_3}(x_1,x_2,x_3) &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^3 \exp\left\{-\tfrac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2 - \tfrac{1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2 \right. \\ &\qquad \left. -\tfrac{1}{2}\left(\frac{x_3-\mu}{\sigma}\right)^2\right\}, \quad -\infty < x_1, x_2, x_3 < \infty. \end{aligned}$$

The transformation

$$\begin{aligned} u &= x_1 - x_3 &&= (x_1-\mu) - (x_3-\mu) \\ v &= x_2 - x_3 &&= (x_2-\mu) - (x_3-\mu) \\ w &= x_1 + x_2 + x_3 - 3\mu &&= (x_1-\mu) + (x_2-\mu) + (x_3-\mu) \end{aligned}$$

is one-to-one and has inverse

$$\begin{aligned} x_1 &= \tfrac{1}{3}(2u - v + w) + \mu \quad \text{i.e.} \quad & x_1 - \mu &= \tfrac{1}{3}(2u - v + w) \\ x_2 &= \tfrac{1}{3}(-u + 2v + w) + \mu \quad \text{i.e.} \quad & x_2 - \mu &= \tfrac{1}{3}(-u + 2v + w) \\ x_3 &= \tfrac{1}{3}(-u - v + w) + \mu \quad \text{i.e.} \quad & x_3 - \mu &= \tfrac{1}{3}(-u - v + w). \end{aligned}$$

/continued overleaf

The Jacobian of the (original) transformation is

$$\begin{aligned} J(u,v,w;x_1,x_2,x_3) &= \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = 2+1=3. \end{aligned}$$

Then the joint p.d.f. of (U,V,W) is

$$\begin{aligned} f_{U,V,W}(u,v,w) &= f_{X,Y,Z}(\tfrac{1}{3}(2u-v+w)+\mu,\ \tfrac{1}{3}(-u+2v+w)+\mu, \\ &\qquad \tfrac{1}{3}(-u-v+w)+\mu)\ |J(x_1,x_2,x_3;u,v,w)|. \end{aligned}$$

Now

$$\begin{aligned} &\{\tfrac{1}{3}(2u-v+w)\}^2 + \{\tfrac{1}{3}(-u+2v+w)\}^2 + \{\tfrac{1}{3}(-u-v+w)\}^2 \\ &= \tfrac{1}{9}\{4u^2+v^2+w^2-4uv+4uw-2vw \\ &\qquad +u^2+4v^2+w^2-4uv-2uw+4vw \\ &\qquad +u^2+v^2+w^2+2uv-2uw-2vw\} \\ &= \tfrac{1}{9}\{6u^2+6v^2+3w^2-6uv\}. \end{aligned}$$

So

$$\begin{aligned} f_{U,V,W}(u,v,w) &= \tfrac{1}{3}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^3 \times \\ &\quad \exp\left\{-\frac{1}{2\sigma^2}\cdot\tfrac{1}{9}(6u^2+6v^2+3w^2-6uv)\right\},\ -\infty < u,v,w < \infty. \end{aligned}$$

This factorises into a function of (U,V) times a function of W. Hence

$$\begin{aligned} \text{p.d.f. of } W &= f_W(w) \propto \exp\left\{-\frac{1}{2(3\sigma^2)}w^2\right\}, \qquad -\infty < w < \infty \\ \Rightarrow \quad f_W(w) &= \frac{1}{\sqrt{2\pi}\sqrt{3}\sigma}\exp\left\{-\frac{1}{2(3\sigma^2)}w^2\right\}, \qquad -\infty < w < \infty \end{aligned}$$

i.e. $W \sim \mathrm{N}(0,3\sigma^2)$, and then

$$f_{U,V}(u,v) = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^2 \exp\left\{-\frac{1}{3\sigma^2}(u^2-uv+v^2)\right\},\ -\infty < u,v < \infty.$$

(ii) (a) Let $\boldsymbol{C}$ be an orthogonal matrix with first row $(a_1,a_2,...,a_n)$ and second row $(b_1,b_2,...,b_n)$.

Let $\boldsymbol{Y} = \boldsymbol{CZ}$ where $\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$, $\boldsymbol{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}$.

Then $Y_1,...,Y_n$ are independent $\mathrm{N}(0,1)$ random variables, and

$$Y_1 = \sum_{i=1}^n a_i Z_i, \qquad Y_2 = \sum_{i=1}^n b_i Z_i.$$

Also

$$\sum_{i=1}^n Y_i^2 = \sum_{i=1}^n Z_i^2.$$

/continued overleaf

So

$$\begin{aligned} W &= \sum_{i=1}^{n} Z_i^2 - \left\{\sum_{i=1}^{n} a_i Z_i\right\}^2 - \left\{\sum_{i=1}^{n} b_i Z_i\right\}^2 \\ &= \sum_{i=1}^{n} Y_i^2 - Y_1^2 - Y_2^2 = \sum_{i=3}^{n} Y_i^2. \qquad (*) \end{aligned}$$

Hence Y_1, Y_2 and W (being a function of $Y_3, ..., Y_n$ only) are independent random
variables, and

$$Y_1 \sim \mathrm{N}(0,1), \quad Y_2 \sim \mathrm{N}(0,1), \quad W \sim \chi^2(n-2)$$

(the last result following from the fact that W is the sum of the squares of $n-2$ independent $\mathrm{N}(0,1)$ random variables – see $(*)$ above.)

(b) Sample mean random variable $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Sample variance random variable $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

$Z_i = (X_i - \mu)/\sigma$ or $X_i = \mu + \sigma Z_i \quad (i = 1, ..., n)$.
So

$$\begin{aligned} \overline{X} &= \mu + \sigma\overline{Z} \qquad \text{(where } \overline{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i) \\ S^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(\mu + \sigma Z_i - \mu - \sigma\overline{Z})^2 = \frac{\sigma^2}{n-1}\sum_{i=1}^{n}(Z_i - \overline{Z})^2. \end{aligned}$$

Since $\overline{Z}$ and $\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \overline{Z})^2$ are independent random variables, it follows that $\overline{X}$ and S^2 are also independent random variables.
Since $\overline{Z} \sim \mathrm{N}(0, \frac{1}{n})$, we have that $\overline{X} \sim \mathrm{N}(\mu, \frac{\sigma^2}{n})$ (because a linear transformation of a normal random variable is also a normal random variable).
Since $\sum_{i=1}^{n}(Z_i - \overline{Z})^2 \sim \chi^2(n-1)$, we have that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.
Now the random variable $T = \frac{Z}{\sqrt{V/r}} \sim t(r)$,
where $Z \sim \mathrm{N}(0,1)$, $V \sim \chi^2(r)$, and Z, V are independent random variables.
So since $\frac{(\overline{X} - \mu)}{\sqrt{\sigma^2/n}} \sim \mathrm{N}(0,1)$ and $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$,
and these random variables are independent, we deduce that

$$\frac{(\overline{X} - \mu)}{\sqrt{\sigma^2/n}} \Bigg/ \sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)} \sim t(n-1)$$

i.e.

$$(\overline{X} - \mu)/\sqrt{S^2/n} \sim t(n-1).$$