SOR201 Solutions to Examples 8

1. (i) (a) The c.d.f. of the uniform distribution on $[0,1]$ is

$$F(x) = x, \qquad 0 \leqslant x \leqslant 1.$$

So the p.d.f. of $X_{(i)}$ is

$$\begin{aligned} f_{(i)}(x) &= \frac{n!}{(i-1)!(n-i)!}\{F(x)\}^{i-1}\{1-F(x)\}^{n-i}f(x), \quad -\infty < x < \infty \\ &= \frac{n!}{(i-1)!(n-i)!}x^{i-1}(1-x)^{n-i}, \qquad 0 \leqslant x \leqslant 1 \\ &= \frac{1}{B(i,n-i+1)}x^{i-1}(1-x)^{n-i}, \qquad 0 \leqslant x \leqslant 1 \end{aligned}$$

i.e. $X_{(i)} \sim$ beta$(i, n-i+1)$.

Then

$$\begin{aligned} \mathrm{E}(X_{(i)}) &= \int_{-\infty}^{\infty} x f_{(i)}(x)dx \\ &= \frac{n!}{(i-1)!(n-i)!}\int_0^1 x^i(1-x)^{n-i}dx \\ &= \frac{n!}{(i-1)!(n-i)!}B(i+1,n-i+1) \\ &= \frac{n!}{(i-1)!(n-i)!}.\frac{\Gamma(i+1)\Gamma(n-i+1)}{\Gamma(n+2)} \\ &= \frac{n!}{(i-1)!(n-i)!}.\frac{i!(n-i)!}{(n+1)!} = \frac{i}{n+1} = p_i \end{aligned}$$

(see lecture notes on the beta distribution).

Also

$$\begin{aligned} \mathrm{E}(X_{(i)}^2) &= \frac{n!}{(i-1)!(n-i)!}\int_0^1 x^{i+1}(1-x)^{n-i}dx \\ &= \frac{n!}{(i-1)!(n-i)!}B(i+2,n-i+1) \\ &= \frac{n!}{(i-1)!(n-i)!}.\frac{(i+1)!(n-i)!}{(n+2)!} \\ &= \frac{i(i+1)}{(n+1)(n+2)}. \end{aligned}$$

So

$$\begin{aligned} \mathrm{Var}(X_{(i)}) &= \frac{i(i+1)}{(n+1)(n+2)} - \left(\frac{i}{n+1}\right)^2 \\ &= \frac{i(n-i+1)}{(n+1)^2(n+2)} \\ &= \frac{p_i q_i}{(n+2)}, \qquad \text{where } q_i = 1 - p_i = \frac{n-i+1}{n+1}. \end{aligned}$$

(b) The joint p.d.f. of $(X_{(i)}, X_{(j)}),\ \ i < j$ is

$$\begin{aligned} f_{(i),(j)}(u,v) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\{F(u)\}^{i-1}\{F(v)-F(u)\}^{j-i-1} \\ &\quad \times\{1-F(v)\}^{n-j}f(u)f(v), \qquad -\infty < u < v < \infty \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}u^{i-1}(v-u)^{j-i-1}(1-v)^{n-j}, \\ &\qquad 0 \leqslant u < v \leqslant 1. \end{aligned}$$

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(c) The joint p.d.f. of $(X_{(1)}, X_{(n)})$ is

$$f_{(1)(n)}(u,v) = \frac{n!}{(n-2)!}(v-u)^{n-2}, \qquad 0 \leqslant u < v \leqslant 1.$$

Let $\quad R = X_{(n)} - X_{(1)},\ \ W = X_{(1)}.$
The transformation $\ r = v-u,\ w = u,\ \ 0 \leqslant u < v \leqslant 1\ $ is one-to-one with inverse $\ u = w,\ v = r + w$. The Jacobian of the (original) transformation is

$$J(r,w;u,v) = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

So the joint p.d.f. of (R,W) is

$$\begin{aligned} f_{R,W}(r,w) &= f_{(1),(n)}(w, r+w)\,|J(u,v;r,w)| \\ &= n(n-1)r^{n-2}\,|-1|\,, \ \ 0 \leqslant w < r+w \leqslant 1, \quad \text{or} \quad 0 \leqslant w \leqslant 1-r. \end{aligned}$$

Then the p.d.f. of R is

$$\begin{aligned} f_R(r) &= \int_0^{1-r} n(n-1)r^{n-2}dw \\ &= n(n-1)r^{n-2}(1-r), \qquad 0 \leqslant r \leqslant 1. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{E}(R) &= n(n-1)\int_0^1 r^{n-1}(1-r)dr \\ &= n(n-1)B(n,2) \\ &= n(n-1).\frac{(n-1)!1!}{(n+1)!} = \frac{n-1}{n+1}. \end{aligned}$$

Also

$$\begin{aligned} \mathrm{E}(R^2) &= n(n-1)\int_0^1 r^{n}(1-r)dr \\ &= n(n-1)B(n+1,2) \\ &= n(n-1).\frac{n!1!}{(n+2)!} = \frac{n(n-1)}{(n+1)(n+2)}. \end{aligned}$$

So

$$\mathrm{Var}(R) = \frac{n(n-1)}{(n+1)(n+2)} - \left(\frac{n-1}{n+1}\right)^2 = \frac{2(n-1)}{(n+1)^2(n+2)}.$$

2. (i) The MGF of X is

$$\begin{aligned} M_X(\theta) &= \mathrm{E}\left(e^{\theta X}\right) = \int_0^\infty e^{\theta x}\lambda e^{-\lambda x}dx \\ &= \int_0^\infty \lambda e^{-(\lambda-\theta)x}dx \\ &= \left[-\frac{\lambda}{\lambda-\theta}e^{-(\lambda-\theta)x}\right]_0^\infty = \frac{\lambda}{\lambda-\theta}, \qquad \theta < \lambda. \end{aligned}$$

Then

$$\begin{aligned} \frac{dM_X}{d\theta} &= \frac{\lambda}{(\lambda-\theta)^2}, \qquad \text{so} \qquad \mathrm{E}(X) = \left[\frac{dM_X}{d\theta}\right]_{\theta=0} = \frac{1}{\lambda}. \\ \frac{d^2M_X}{d\theta^2} &= \frac{2\lambda}{(\lambda-\theta)^2}, \qquad \text{so} \qquad \mathrm{E}(X^2) = \left[\frac{d^2M_X}{d\theta^2}\right]_{\theta=0} = \frac{2}{\lambda^2} \end{aligned}$$

$$\text{and} \quad \mathrm{Var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

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(ii) (a) $M_W(\theta) = M_{X_1+\cdots+X_n}(\theta) = M_{X_1}(\theta)...M_{X_n}(\theta) = \left(\frac{\lambda}{\lambda-\theta}\right)^n, \quad \theta < \lambda.$

The MGF of the gamma(α, λ) distribution is

$$\begin{aligned} M_U(\theta) &= \int_0^\infty e^{\theta u}.\frac{\lambda^\alpha u^{\alpha-1}e^{-\lambda u}}{\Gamma(\alpha)}du \\ &= \int_0^\infty \frac{\lambda^\alpha u^{\alpha-1}e^{-(\lambda-\theta)u}}{\Gamma(\alpha)}du \qquad [\text{set} \quad t=(\lambda-\theta)u,\ \lambda>\theta] \\ &= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)}\left(\frac{t}{\lambda-\theta}\right)^{\alpha-1} e^{-t}\frac{1}{(\lambda-\theta)}dt \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda-\theta)^\alpha}\int_0^\infty t^{\alpha-1}e^{-t}dt \\ &= \frac{\lambda^\alpha}{(\lambda-\theta)^\alpha} = \left(\frac{\lambda}{\lambda-\theta}\right)^\alpha, \quad \theta<\lambda. \end{aligned}$$

Hence $\quad W = X_1 + \cdots + X_n \sim$ gamma(n, λ).

(b) The MGF of $\overline{X}$ is

$$\begin{aligned} M_{\overline{X}}(\theta) &= \mathrm{E}\left(e^{\theta\overline{X}}\right) \\ &= \mathrm{E}\left(e^{\frac{\theta}{n}\sum_i X_i}\right) \\ &= M_W(\theta/n) \\ &= \left(\frac{\lambda}{\lambda-\theta/n}\right)^n = \left(\frac{n\lambda}{n\lambda-\theta}\right)^n, \quad \theta/n<\lambda. \end{aligned}$$

So $\quad \overline{X} \sim$ gamma$(n, n\lambda)$.

(c) We have

$$\begin{aligned} M_{U_1+U_2}(\theta) &= M_{U_1}(\theta)M_{U_2}(\theta) \\ &= \left(\frac{\lambda}{\lambda-\theta}\right)^{\alpha_1}.\left(\frac{\lambda}{\lambda-\theta}\right)^{\alpha_2}, \qquad \theta<\lambda \\ &= \left(\frac{\lambda}{\lambda-\theta}\right)^{\alpha_1+\alpha_2}. \end{aligned}$$

So $\quad U_1 + U_2 \sim$ gamma$(\alpha_1+\alpha_2, \lambda)$.

3. (i) The MGF of Y is

$$\begin{aligned} M_Y(\theta) &= \mathrm{E}\left(e^{\theta Y}\right) = \int_a^b \frac{e^{\theta y}}{b-a}dy \\ &= \left[\frac{e^{\theta y}}{\theta(b-a)}\right]_{y=a}^{y=b} \\ &= \frac{e^{\theta b}-e^{\theta a}}{\theta(b-a)}. \end{aligned}$$

Expanding the exponentials:

$$\begin{aligned} M_Y(\theta) = &\left\{1+\theta b+\tfrac{1}{2!}(\theta b)^2+\tfrac{1}{3!}(\theta b)^3+\cdots\right. \\ &\left. -1-\theta a-\tfrac{1}{2!}(\theta a)^2-\tfrac{1}{3!}(\theta a)^3-\cdots\right\}/\{\theta(b-a)\} \end{aligned}$$

/continued overleaf

In this series expansion, the coefficient of θ is $\dfrac{\frac{1}{2!}b^2 - \frac{1}{2!}a^2}{b-a} = \frac{1}{2}(b+a)$, so

$$\mathrm{E}(X) = \text{coefficient of } \theta/1! = \tfrac{1}{2}(a+b).$$

The coefficient of θ^2 is $\dfrac{\frac{1}{3!}b^3 - \frac{1}{3!}a^3}{b-a} = \frac{1}{6}(b^2+ab+a^2)$, so

$$\mathrm{E}(X^2) = \text{coefficient of } \theta^2/2! = \tfrac{1}{3}(b^2+ab+a^2)$$

and then

$$\begin{aligned}\mathrm{Var}(X) &= \tfrac{1}{3}(b^2+ab+a^2) - \tfrac{1}{4}(a+b)^2 \\ &= \tfrac{1}{12}(b^2-2ab+a^2) = \tfrac{1}{12}(b-a)^2.\end{aligned}$$

[Note: the calculation of moments via $(\frac{dM}{d\theta})_{\theta=0}$ and $(\frac{d^2M}{d\theta^2})_{\theta=0}$ is more difficult here.]

(ii) (a) If $X \sim$ uniform $[0,1]$, then from part (i)

$$M_X(\theta) = \frac{e^\theta - 1}{\theta} \qquad [\text{special case } a=0, b=1].$$

(b) Since $X_1, ..., X_n$ are independent,

$$M_{X_1+\cdots+X_n}(\theta) = \left\{\frac{e^\theta - 1}{\theta}\right\}^n.$$

Then

$$\begin{aligned}M_{\overline{X}}(\theta) &= \mathrm{E}\left(e^{\theta\overline{X}}\right) \\ &= \mathrm{E}\left(e^{(\theta/n)(X_1+\cdots+X_n)}\right) \\ &= M_{X_1+\cdots+X_n}(\theta/n) \\ &= \left\{\frac{e^{\theta/n}-1}{\theta/n}\right\}^n.\end{aligned}$$

(c) It follows that

$$\begin{aligned}\log_e M_{\overline{X}}(\theta) &= n\log_e\left\{\frac{e^{\theta/n}-1}{\theta/n}\right\} \\ &= n\log_e\left\{\frac{1+(\theta/n)+\frac{1}{2!}(\theta/n)^2+\frac{1}{3!}(\theta/n)^3+\cdots-1}{\theta/n}\right\} \\ &= n\log_e\{1+\langle\tfrac{1}{2}(\theta/n)+\tfrac{1}{6}(\theta/n)^2+o(\tfrac{1}{n^2})\rangle\} \\ &= n\{\quad\langle\tfrac{1}{2}(\theta/n)+\tfrac{1}{6}(\theta/n)^2+o(\tfrac{1}{n^2})\rangle \\ &\qquad -\tfrac{1}{2}\langle\ldots\ldots\ldots\text{ditto}\ldots\ldots\ldots\rangle^2 \\ &\qquad +\tfrac{1}{3}\langle\ldots\ldots\ldots\text{ditto}\ldots\ldots\ldots\rangle^3-\cdots\}\end{aligned}$$

(provided $|\langle\text{....ditto....}\rangle| < 1$,
which is satisfied for a given value of θ
by choosing n sufficiently large)

$$\begin{aligned}&= \tfrac{1}{2}\theta + \tfrac{1}{6}\theta^2.\tfrac{1}{n} - \tfrac{1}{2}\{\tfrac{1}{2}.\tfrac{\theta}{n}\}^2 n + o(\tfrac{1}{n}) \\ &= \tfrac{1}{2}\theta + \tfrac{1}{24}\theta^2.\tfrac{1}{n} + o(\tfrac{1}{n}).\end{aligned}$$

So

$$\begin{aligned}M_{\overline{X}}(\theta) &= \exp\{\tfrac{1}{2}\theta + \tfrac{1}{24n}.\theta^2 + o(\tfrac{1}{n})\} \\ &\approx \exp\{\tfrac{1}{2}\theta + \tfrac{1}{2}.\tfrac{1}{12n}\theta^2\} \qquad \text{when } n \text{ is large.}\end{aligned}$$

So

$$\overline{X} \overset{\text{approx}}{\sim} \mathrm{N}(\tfrac{1}{2}, \tfrac{1}{12n}) \quad \text{when } n \text{ is large.}$$

(Since $\mathrm{E}(X_i) = \frac{1}{2}$, $\mathrm{Var}(X_i) = \frac{1}{12}$,
then $\mathrm{E}(\overline{X}) = \frac{1}{2}$, $\mathrm{Var}(\overline{X}) = \frac{1/12}{n} = \frac{1}{12n}$.)

4. (i)

$$M_{X_i}(\theta) = \mathrm{E}\left(e^{\theta X_i}\right) = \exp(\mu_i\theta + \tfrac{1}{2}\sigma_i^2\theta^2).$$

$$\begin{aligned} M_{a_iX_i}(\theta) &= \mathrm{E}\left(e^{\theta a_i X_i}\right) = \mathrm{E}\left(e^{(a_i\theta)X_i}\right) \\ &= M_{X_i}(a_i\theta) = \exp(\mu_i a_i\theta + \tfrac{1}{2}\sigma_i^2 a_i^2\theta^2) \\ &= \exp(a_i\mu_i\theta + \tfrac{1}{2}a_i^2\sigma_i^2\theta^2). \end{aligned}$$

Then

$$\begin{aligned} M_W(\theta) &= M_{\sum_i \sigma_i a_i X_i}(\theta) = M_{a_1X_1}.........M_{a_nX_n}(\theta) \qquad \text{[independence]} \\ &= \exp(a_1\mu_1\theta + \tfrac{1}{2}a_1^2\sigma_1^2\theta^2).........\exp(a_n\mu_n\theta + \tfrac{1}{2}a_n^2\sigma_n^2\theta^2) \\ &= \exp\left\{\left(\sum_{i=1}^n a_i\mu_i\right)\theta + \tfrac{1}{2}\left(\sum_{i=1}^n a_i^2\sigma_i^2\right)\theta^2\right\} \end{aligned}$$

So $$W \sim \mathrm{N}\left(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2\right).$$

If $X_i \sim \mathrm{N}(\mu, \sigma^2) \quad (i = 1, ..., n)$, and writing $\overline{X}$ in the form $\overline{X} = \sum_{i=1}^n (\frac{1}{n})X_i$, the above result gives

$$\overline{X} \sim \mathrm{N}\left(\sum_{i=1}^n \left(\tfrac{1}{n}\right)\mu, \sum_{i=1}^n \left(\tfrac{1}{n}\right)^2\sigma^2\right) = \mathrm{N}(\mu, \sigma^2/n).$$

(ii) $M_{X_1,X_2}(\theta_1,\theta_2) = \mathrm{E}\left(e^{\theta_1X_1+\theta_2X_2}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\theta_1x_1+\theta_2x_2}f(x_1,x_2)dx_1dx_2$

provided the integral exists in some region $|\theta_1| < \theta_{10},\ |\theta_2| < \theta_{20}$.

(a) $M_{X_1,X_2}(\theta_1,\theta_2) = \exp\{\mu_1\theta_1 + \mu_2\theta_2 + \frac{1}{2}(\sigma_1^2\theta_1^2 + 2\rho\sigma_1\sigma_2\theta_1\theta_2 + \sigma_2^2\theta_2^2)\}$.

$\dfrac{\partial M_{X_1,X_2}}{\partial\theta_1} = \exp\{\text{...ditto...}\}(\mu_1 + \sigma_1^2\theta_1 + \rho\sigma_1\sigma_2\theta_2)$.

So $\mathrm{E}(X_1) = \left[\dfrac{\partial M_{X_1,X_2}}{\partial\theta_1}\right]_{\theta_1=\theta_2=0} = \mu_1.$

Similarly, $\mathrm{E}(X_2) = \mu_2$.

Also, $\dfrac{\partial^2 M_{X_1,X_2}}{\partial\theta_1^2} = \exp\{\text{...ditto...}\}(\mu_1+\sigma_1^2\theta_1+\rho\sigma_1\sigma_2\theta_2)^2 + \exp\{\text{...ditto...}\}\sigma_1^2$.

So $\mathrm{E}(X_1^2) = \left[\dfrac{\partial^2 M_{X_1,X_2}}{\partial\theta_1^2}\right]_{\theta_1=\theta_2=0} = \mu_1^2 + \sigma_1^2$

and $\mathrm{Var}(X_1) = (\mu_1^2 + \sigma_1^2) - \mu_1^2 = \sigma_1^2$.

Similarly, $\mathrm{Var}(X_2) = \sigma_2^2$.

Finally,

$$\begin{aligned} \frac{\partial^2 M_{X_1,X_2}}{\partial\theta_1\partial\theta_2} &= \exp\{\text{...ditto...}\}(\mu_2 + \rho\sigma_1\sigma_2\theta_1 + \sigma_2^2\theta_2).(\mu_1 + \sigma_1^2\theta_1 + \rho\sigma_1\sigma_2\theta_2) \\ &\quad + \exp\{\text{...ditto...}\}(\rho\sigma_1\sigma_2). \end{aligned}$$

So $\mathrm{E}(X_1X_2) = \left[\dfrac{\partial^2 M_{X_1,X_2}}{\partial\theta_1\partial\theta_2}\right]_{\theta_1=\theta_2=0} = \mu_2\mu_1 + \rho\sigma_1\sigma_2.$

Hence $\mathrm{Cov}(X_1,X_2) = \mathrm{E}(X_1X_2) - \mathrm{E}(X_1)\mathrm{E}(X_2) = \rho\sigma_1\sigma_2$

and $\rho(X_1,X_2) = \dfrac{\mathrm{Cov}(X_1,X_2)}{\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)}} = \dfrac{\rho\sigma_1\sigma_2}{\sqrt{\sigma_1^2\sigma_2^2}} = \rho.$

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(b) We have

$$\begin{aligned} M_{a_1X_1+a_2X_2}(\theta) &= \mathrm{E}\left[e^{(a_1X_1+a_2X_2)\theta}\right] \\ &= \mathrm{E}\left[e^{(a_1\theta)X_1+(a_2\theta)X_2}\right] \\ &= M_{X_1,X_2}(a_1\theta, a_2\theta) \\ &= \exp\{a_1\mu_1\theta + a_2\mu_2\theta + \tfrac{1}{2}(a_1^2\sigma_1^2 + 2a_1a_2\rho\sigma_1\sigma_2 + a_2^2\sigma_2^2)\theta^2\} \\ &= \text{MGF of } \; \mathrm{N}(a_1\mu_1 + a_2\mu_2, a_1^2\sigma_1^2 + 2a_1a_2\rho\sigma_1\sigma_2 + a_2^2\sigma_2^2). \end{aligned}$$

So $a_1X_1 + a_2X_2 \sim \mathrm{N}(a_1\mu_1 + a_2\mu_2, a_1^2\sigma_1^2 + 2a_1a_2\rho\sigma_1\sigma_2 + a_2^2\sigma_2^2)$.

(iii) (a) If $S_n = X_1 + \cdots + X_n$, where $X_1, ..., X_n$ are i.i.d. Poisson(1) random variables, then

$$S_n \sim \text{Poisson}(n).$$

(b) The Poisson(1) distribution has mean 1 and standard deviation 1.
So by the central limit theorem

$$\frac{S_n - n.1}{\sqrt{n}.1} \overset{a}{\sim} \mathrm{N}(0,1),$$

i.e. the c.d.f. of $\dfrac{S_n - n}{\sqrt{n}} \to$ the c.d.f. of $\mathrm{N}(0,1)$ as $n \to \infty$.

(c) We have

$$\begin{aligned} \mathrm{P}(S_n \leqslant n) &= \mathrm{P}(S_n - n \leqslant 0) \\ &= \mathrm{P}\left(\frac{S_n - n}{\sqrt{n}} \leqslant 0\right) \to \mathrm{P}(Z \leqslant 0) \text{ as } \; n \to \infty \\ &= \tfrac{1}{2}. \qquad [\text{since } \; Z \sim \mathrm{N}(0,1)] \end{aligned}$$

But since $S_n \sim \text{Poisson}(n)$,

$$\mathrm{P}(S_n = j) = \frac{n! e^{-n}}{j!}, \; j = 0, 1, ..., \quad \text{and} \quad \mathrm{P}(S_n \leqslant n) = \sum_{j=0}^{n} \frac{n^j e^{-n}}{j!}.$$

Hence $e^{-n}(1 + n + \dfrac{n^2}{2!} + \cdots + \dfrac{n^n}{n!}) \to \tfrac{1}{2}$ as $n \to \infty$.