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Non-Markovian decoherence: complete positivity and decomposition

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In this work, the non-Markovian decoherence is considered in two ways. Firstly, an effective Hamiltonian approach is demonstrated to investigate the decoherence of a quantum system in a non-Markovian environment, in which complete positivity of the reduced dynamics is achieved. This method uses the notion of an effective environment, that is a subsystem of the environment that causes the decoherence. Secondly, the evolution of the system and environment is decomposed, thus partially illuminating how they would interact given that memory effects are allowed. It should be noted that beam splitters and rotators are sufficient to explain this decomposition.

1. Introduction

Decoherence or loss of coherence within a quantum system is an issue that cannot be ignored, for it is the very reason why quantum-mechanical effects are not seen in everyday life. Any non-trivial quantum system is open; thus dissipation inevitably happens and so coherence is lost. Decoherence is due to the interaction of the quantum system with its surrounding environment. In general the environment is assumed to be large in comparison with the system of interest. Thus the following approximations are often made.

(i) The effect of the system–environment interaction is negligible on the state of the environment. That is, if the environment is altered by the interaction, it will quickly decay or relax back to its original state before the next system–environment interaction occurs. Therefore, the characteristic times are such that $\tau_R \ll \tau_D$ where τ_R is the relaxation time of the environment and τ_D is the decoherence time due to the system–environment interaction.

(ii) The coupling between the environment and system is weak so that the initial uncorrelated state of the system and environment is only altered by orders of the interaction Hamiltonian. However, within the derivation of the master equation, third-order and higher terms are neglected. Thus only the state of the system is affected by the interaction whilst the environment's state remains essentially unaffected.

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(iii) The future evolution of the system depends only on the present and not the past history of the system. Together, these form the Born–Markov approximation [1].

Beyond these approximations the decoherence is termed non-Markovian. By allowing $\tau_R \gtrsim \tau_D$ it is now feasible for the environment to incorporate memory effects. That is, once the environment and system have interacted, a second interaction can occur before the environment can relax back to its original state. Thus the former interaction has an effect on the latter interaction because the environment is able to memorize a part of the system's information during τ_R, which in turn affects the system during the latter interaction still within the time τ_R. In this work the environment will be of a non-Markovian nature. Hence no assumptions are made about the coupling strength, nor about the characteristic times τ_D and τ_R.

It is important to derive a completely positive master or Fokker–Planck equation to describe the evolution of the state. Otherwise there will only be a probability that the correct physical dynamics will be seen [2]. This, however, has proven difficult to achieve when dealing with an open quantum system. In section 2, we shall illustrate a method that will guarantee complete positivity of the reduced dynamics and hence the correct physical dynamic description of the system.

In order to understand the mechanism of decoherence, a decomposition of the evolution between the system and its environment would be advantageous. Thus, in understanding it, more effective ways to overcome it may be possible or, as has been suggested recently, using decoherence as a resource. For a non-Markovian environment, it will be shown that a decomposition is possible for the interaction Hamiltonian (20) where a finite number N of environmental bosonic modes are considered. Lie algebra indicates that this decomposition should indeed exist. Thus, for a system–environment in resonance, the evolution is decomposable with only rotators and beam splitters required. This will be illustrated in section 3.

2. Complete positivity

The evolution of a total density matrix is usually determined by the derivation of the master equation. Consider a system–environment interaction, where the total Hamiltonian in the Schrödinger picture (denoted by the superscript s) is of the form

$$\hat{H}_T^s = \hat{H}_0^s + \hat{H}_I^s, \tag{1}$$

where $\hat{H}_0^s = \hat{H}_S^s + \hat{H}_R^s$ is the free-field Hamiltonian of the system and environment and $\hat{H}_I^s$ is the interaction Hamiltonian due to the system–environment interaction. A typical form for the interaction Hamiltonian may be given by

$$\hat{H}_I^s = \sum_i \hat{S}_i \otimes \hat{R}_i, \tag{2}$$

where $\hat{S}_i$ is the operator acting on the system and $\hat{R}_i$ is due to the environment interaction with the system [3]. This operator $\hat{R}_i$ may take various forms depending on the type and nature of the interaction. For instance, for N harmonic oscillators, $\hat{R}_i = \sum_j g_{ij}\hat{b}_j^\dagger + g_{ij}^*\hat{b}_j$, whilst those bosonic modes unaffected by the interaction will evolve freely governed by $\hat{H}_R^s = \sum_j \omega_{b_j}\hat{b}_j^\dagger\hat{b}_j$. g_{ij} is the coupling strength of interaction involving that particular bosonic mode and ω_{b_j} is the frequency of that mode.

The master equation can be derived in a number of different ways, although the derivation based on projection operators will be used here [4]. The quantum Liouville equation for a system coupled to the environment for the density operator of the total system in the Schrödinger picture is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}(\tau) = \frac{1}{\mathrm{i}\hbar}\Big[\hat{H}_{\mathrm{T}}^{\mathrm{s}}, \hat{\rho}_{\mathrm{T}}^{\mathrm{s}}(\tau)\Big] = \frac{1}{\mathrm{i}\hbar}\mathcal{L}_{\mathrm{T}}\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}(\tau) \tag{3}$$

where $\mathcal{L}_{\mathrm{T}} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}}$, $(\mathcal{L}_0 = \mathcal{L}_S + \mathcal{L}_R)$ is the Liouville operator of the total system, and $\mathcal{L}_0$ and $\mathcal{L}_{\mathrm{I}}$ are in one-to-one correspondence to the Hamiltonians. The environment is initially assumed to be in a thermal state

$$\hat{\rho}_R^s = \frac{\exp\Big(-\hat{H}_R^{\mathrm{s}}/kT\Big)}{\mathrm{Tr}\Big[\exp\Big(-\hat{H}_R^s/kT\Big)\Big]} \tag{4}$$

such that $\mathcal{L}_R\hat{\rho}_R^{\mathrm{s}} = 0$ is satisfied. We are interested in the reduced dynamics of the system, that is $\hat{\rho}_S^{\mathrm{s}} = \mathrm{Tr}_{\mathrm{R}}(\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}(\tau))$ but this can be seen as a projection of the operator $\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}(\tau)$ by the projection operators $\mathcal{P}$ and $\mathcal{Q}$ where

$$\mathcal{P}\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}(\tau) = \mathrm{Tr}_R(\hat{\rho}_T^s(\tau)) \otimes \hat{\rho}_R^s, \quad \mathcal{Q} = 1 - \mathcal{P}. \tag{5}$$

Thus the reduced density matrix of the system is given by $\hat{\rho}_S^{\mathrm{s}} = \mathrm{Tr}_R(\mathcal{P}\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}(\tau))$ and $\hat{\rho}_{\mathrm{T}}^{\mathrm{s}} = \mathcal{P}\hat{\rho}_{\mathrm{T}}^{\mathrm{s}} + \mathcal{Q}\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}$ resulting in two coupled equations from equation (3). Using the condition that $\mathcal{P}\mathcal{L}_I\mathcal{P} = 0$ where this represents no energy shift for the system, a solution for $\mathcal{Q}\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}$ may be obtained and on substituting back into $\mathcal{P}\hat{\rho}_{\mathrm{T}}^{\mathrm{s}}$ gives the master equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\rho}_S(\tau) = \mathcal{C}(\tau)\hat{\rho}_S(\tau) \tag{6}$$

where $\hat{\rho}_S(\tau) = \exp(i\tau\mathcal{L}_S)\ \hat{\rho}_S^{\mathrm{s}}(\tau)$ is the reduced density matrix in the interaction picture and $\mathcal{C}(\tau)$ is the generalized collision operator.

Within the weak-coupling approximation (short timescale) and by expanding the time-dependent operators by a complete set of Hermitian operators C, such that

$$\hat{S}_i(\tau) = \sum_{j=1}^{d^2-1} c_{ij}(\tau)\hat{S}_j, \tag{7}$$

where d is the dimension of the system Hilbert space and the time dependence enters the coefficients $c_{ij}(\tau)$ which can be determined via the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{S}_i = \frac{1}{\mathrm{i}\hbar}\Big[\hat{H}_S, \hat{S}_i(\tau)\Big], \tag{8}$$

the collision operator can then be written as

$$C(\tau)\hat{\rho}_S(\tau) = \sum_{ij=1}^{d^2-1}\Big\{\gamma_{ij}(\tau)[\hat{S}_j\hat{\rho}_S(\tau), \hat{S}_i] + \gamma_{ji}^*(\tau)[\hat{S}_j, \hat{\rho}_S(\tau)\hat{S}_i]\Big\}, \tag{9}$$

where

$$\gamma_{ij}(\tau) = \int_0^\tau \mathrm{d}\tau' \sum_{kl} c_{ki}(\tau)\tilde{\chi}_{kl}(\tau - \tau')c_{lj}(\tau'),$$

$$\gamma^*_{ji}(\tau) = \int_0^\tau \mathrm{d}\tau' \sum_{kl} c_{ki}(\tau')\tilde{\chi}_{kl}(\tau' - \tau)c_{lj}(\tau),$$

$$\tilde{\chi}_{kl}(\tau - \tau') = \mathrm{Tr}_R\Big[\hat{R}_k(\tau)\hat{R}_l(\tau')\hat{\rho}^s_R\Big],$$

$$\hat{R}_k(\tau) = \exp(\mathrm{i}\tau\mathcal{L}_R)\hat{R}_k.$$

Thus a time convolutionless form of the master equation for the given system–environment interaction has been obtained, which is valid only within short timescales owing to the approximations made.

Secondly, now introduce the idea of effective environmental variables. The environment contains a normally infinite number of degrees of freedom. This can be translated into an effective environment containing all physical properties of the complete environment, but mathematically the infinite degrees of freedom are now finite, which is easier to deal with. This may be achieved via the Fourier transformation. Using either the complete or effective environmental variables will result in the same master or Fokker–Planck equation, hence the reduced dynamics are consistent. For instance, a Markovian thermal environment is reducible to a collective single-mode boson field in a thermal state [5]. Additionally, effective variables have the property that, as the interaction time τ passes, the dynamic behaviour (if present) can be absorbed into time-dependent coupling constants, thus leaving the effective variables stationary. A complete picture of the decoherence can therefore be formed given these coupling constants and the effective variables [6].

To derive the effective master equation, now let the interaction Hamiltonian within the interaction picture be of the form

$$\hat{H}_\mathrm{I}(\tau) = \sum_i \hat{S}_i \otimes \hat{D}_i(\tau), \tag{10}$$

where again $\hat{S}_i$ is the operator acting on the system and $\hat{D}_i(\tau) = \sum_j \lambda_{ij}(\tau)\hat{E}_j$ given that $\hat{E}_j$ is the operator of the effective environment E and $\lambda_{ij}(\tau)$ is the time-dependent coupling constant. Because of the very nature of the effective environment, initial conditions on the environment cannot be known. For instance, an environment may be in thermal equilibrium initially but, on transformation to the effective environment variables, the state need no longer be a thermal state. Thus, to gain the required boundary conditions for the (effective) master equation, a comparison of the master equations from both approaches is made. The reasoning can be seen as follows: both are representing the same system, and as such within the short time limit they should be identical. Thus, from the standard derivation, positive reduced dynamics were obtained only in this short time limit, whilst the reduced dynamics from the effective master equation are true for all time. Comparison yields the necessary conditions on the reduced dynamics from the master equation from the effective environment approach to be positive for all times, which are that

$$\mathrm{Tr}_R\hat{D}_i(\tau)\hat{\rho}_R(0) = 0 \tag{11}$$

which is similar to the requirement that $\mathcal{P}\mathcal{L}_I\mathcal{P} = 0$ and

$$\int_0^\tau \mathrm{d}\tau' \,\mathrm{Tr}_R\Big[\hat{D}_i(\tau)\hat{D}_j(\tau')\hat{\rho}_R(0)\Big] \approx \gamma_{ij}(\tau). \tag{12}$$

It is required that both equation (11) and equation (12) are satisfied for the positivity of the reduced dynamics of the system to be guaranteed. That is, these are the conditions required for both of the master equations to be identical on the short timescale (weak coupling limit).

The positivity of the reduced dynamics of the system is now guaranteed because of the following facts.

(i) Let $\boldsymbol{V}$ be a vector space of the set of bounded operators with inner product defined by the Hilbert–Schmidt norm: $\langle\langle \hat{\boldsymbol{x}}|\hat{\boldsymbol{y}}\rangle\rangle = \mathrm{Tr}\,(\hat{\boldsymbol{x}}^\dagger \hat{\boldsymbol{y}})$. Letting $|\hat{\boldsymbol{x}}\rangle\rangle$ be the ket vector of the operator $\hat{\boldsymbol{x}}$ in analogy to the ket state and similarly for the bra vector. Also let script letters denote superoperators in the Hilbert–Schmidt space [7].

(ii) The superoperator $\mathcal{S}$ is the linear map on $\boldsymbol{V}$ onto itself, is Hermitian if $\mathcal{S}_{ab,\,cd} = \mathcal{S}^*_{cd,\,ab}$ and positive if $\langle\langle \hat{\boldsymbol{v}}|\mathcal{S}|\hat{\boldsymbol{v}}\rangle\rangle$ is real and positive $\forall\, \hat{\boldsymbol{v}} \in \boldsymbol{V}$.

(iii) If $\mathcal{S}$ is Hermitian, it will satisfy $\mathcal{S}|\boldsymbol{v}\rangle\rangle = s|\boldsymbol{v}\rangle\rangle$ where $|\boldsymbol{v}\rangle\rangle$ is the right eigenvector with eigenvalue s. It must be noted that, if $\mathcal{S}$ is positive, all eigenvalues will be positive, and vice versa.

(iv) Defining the partial transposition on a superoperator by $\mathcal{S}^{\mathrm{PT}}$ as $\mathcal{S}^{\mathrm{PT}}_{ab,\,cd} = \mathcal{S}_{ac,\,bd}$, then it is Hermitian preserving if $\mathcal{S}(\boldsymbol{x}^\dagger) = \mathcal{S}(\boldsymbol{x})^\dagger$ for any $\boldsymbol{x} \in \boldsymbol{V}$, and $\mathcal{S}^{\mathrm{PT}}$ is Hermitian iff $\mathcal{S}$ is Hermitian preserving.

(v) The following statements are equivalent for a superoperator $\mathcal{S}$.

  (I) $\mathcal{S}$ is completely positive.
  (II) $\mathcal{S}$ has a Kraus representation.
  (III) $\mathcal{S}^{\mathrm{PT}}$ is positive.

The equivalence can be reasoned as follows: For (I) $\Longleftrightarrow$ (II) see [8, 9]. For (II) $\Longrightarrow$ (III), the Kraus representation for the matrix elements of $\mathcal{S}$ is given by $\mathcal{S}_{ab,\,cd} = \sum_\mu K^{ac}_\mu K^{bd*}_\mu$ such that

$$\mathcal{S}^{\mathrm{PT}}_{ab,\,cd} = \mathcal{S}_{ac,\,bd} = \sum_\mu K^{ab}_\mu K^{cd*}_\mu. \tag{13}$$

Thus

$$\begin{aligned}\langle\langle \boldsymbol{v}|\mathcal{S}^{\mathrm{PT}}|\boldsymbol{v}\rangle\rangle &= \sum_{abcd}\sum_\mu v^*_{ab}K^{ab}_\mu K^{cd*}_\mu v_{cd} \\ &= \sum_\mu X_\mu X^*_\mu > 0,\end{aligned} \tag{14}$$

where $X_\mu = \sum_\mu v^*_{ab}K^{ab}_\mu$ and so $\mathcal{S}^{\mathrm{PT}}$ is positive. (III) $\Longleftrightarrow$ (II), given that $\mathcal{S}^{\mathrm{PT}}$ is positive, it is Hermitian and so has positive eigenvalues. That is,

$$\mathcal{S}^{\mathrm{PT}}_{ab,\,cd} = \sum_\alpha s_\alpha \langle\langle e_{ab}|v_\alpha\rangle\rangle\langle\langle v_\alpha|e_{cd}\rangle\rangle, \tag{15}$$

where $|v_\alpha\rangle\rangle$ are normalized eigenvectors and $\{e_{ij}\}$ form an orthonormal basis for the vector space. As s_α is positive, it is possible to define a matrix

$$\tilde{K}^{ab}_\alpha = \sqrt{s_\alpha}\langle\langle e_{ab}|v_\alpha\rangle\rangle \tag{16}$$

such that the superoperator $\mathcal{S}$ now has a Kraus representation

$$\mathcal{S}_{ab,\,cd} = \mathcal{S}^{\mathrm{PT}}_{ac,\,bd} = \sum_{\alpha} \big(\tilde{K}^{ac}_{\alpha}\big)\big(\tilde{K}^{bd}_{\alpha}\big)^{*}. \tag{17}$$

Although this Kraus representation is not unique, all forms can be generated by 'unitary remixing' of the canonical set with the eigenvector $\boldsymbol{s}'$ extended by zeros [10].

Given that conditions (11) and (12) are now satisfied such that the reduced dynamics from the effective environmental approach are correct, the quantum Liouville equation for the composite system may be solved [11]. The evolution of the initial state can now be determined by $\hat{\rho}_{\mathrm{T}}(\tau) = \hat{U}_{\mathrm{I}}(\tau)\hat{\rho}_{\mathrm{T}}(0)\hat{U}^{\dagger}_{\mathrm{I}}(\tau)$ and by tracing over E, $\hat{\rho}_S(\tau)$ is obtained. It is sometimes worthwhile to let $\lambda_{ij}(\tau) = \lambda(\tau)g_{ij}$ or $\hat{D}_i(\tau) = \lambda(\tau)\hat{D}_i$ so that $\hat{U}_{\mathrm{I}}(\tau) = \exp\big[-\mathrm{i}\int_0^{\tau} \mathrm{d}\tau'\ \lambda(\tau') \sum_j \hat{S}_j \otimes \hat{D}_j\big]$. By defining an orthonormal basis set of E that diagonalizes $\hat{\rho}_R(0)$, then the reduced dynamics of the system have a Kraus [8] representation,

$$\hat{\rho}_S(\tau) = \mathcal{S}(\tau)\hat{\rho}_S(0) = \sum_{n,m} \hat{K}_{nm}(\tau)\hat{\rho}_S(0)\hat{K}^{\dagger}_{nm}(\tau) \tag{18}$$

where $\hat{K}_{nm}(\tau) = p_m\ {}_r\langle n|\hat{U}_{\mathrm{I}}(\tau)|m\rangle_r$. By the above reasoning, this implies the complete positivity of the evolution superoperator $\mathcal{S}$. This outlines how a superoperator solution of the effective variable equivalent of equation (6) can be achieved which is completely positive for all times, owing to the effective variable approach.

3. Decomposition of the interaction Hamiltonian

Until now the dynamic behaviour of a system–environment interaction has been dealt with by considering the reduced dynamics of the composite system. By tracing over the environmental variables all insight into the environment is lost. To understand decoherence, one must first understand how the environment influences the system. To address this issue fully, questions such as the following need to be asked. What is the actual mechanism by which the coherence of a system is lost? How does decoherence take place? What is the environment actually like? To try to answer these questions the Markovian approximation is often made in an attempt to deal with any system–environment interaction. Nonetheless, within the Markovian approximation the whole story is not revealed because of the approximations enforced. As a result, Markovian decoherence studies are considered inadequate with regard to solid-state systems including a photonic band-gap material and a quantum dot.

Within the Markovian approximation, the infinite degrees of freedom of the environment may be achieved by modelling the environment by an infinite array of beam splitters each with a thermal input field [12]. This essentially brings thermal noise into the system. However, the dynamics of this infinite array of beam splitters can be achieved equivalently by a single collective (effective) mode in thermal equilibrium. Beyond the Markovian approximation, the environment can no longer be represented by an array of beam-splitter operators, as by the Campbell–Baker–Hausdorff theorem, $\exp(\hat{a}\hat{b}^{\dagger}_i - \hat{a}^{\dagger}\hat{b}_i)\exp(\hat{a}\hat{b}^{\dagger}_j - \hat{a}^{\dagger}\hat{b}_j) \neq \exp(\hat{a}(\hat{b}^{\dagger}_i + \hat{b}^{\dagger}_j) - \hat{a}^{\dagger}(\hat{b}_i + \hat{b}_j))$ as $[\hat{a}\hat{b}^{\dagger}_i - \hat{a}^{\dagger}\hat{b}_i,\ \hat{a}\hat{b}^{\dagger}_j - \hat{a}^{\dagger}\hat{b}_j] = \hat{b}^{\dagger}_i\hat{b}_j - \hat{b}^{\dagger}_j\hat{b}_i \neq 0$ given that $i \neq j$.

With these issues in mind, we shall investigate a system–environment interaction beyond the Markovian approximation and extend the approach in [5] to incorporate N modes of the environment with natural frequencies ω_{b_m}, that influence the system with arbitrary coupling constants G_m. By observing how individual bosonic operators evolve within such an interaction, the interaction Hamiltonian will be decomposed into local operators of rotators and beam splitters so that the bosonic operators will evolve in the same way. Consequently, the decomposed evolution operator may be applied to any system of interest. It must be noted that this (most likely) is only one possible way of representing the non-Markovian evolution operator and is in no way intended as a general or only possible result. Nonetheless, the physical interpretation of the decomposition is sound with current understanding of non-Markovian decoherence.

3.1 *Non-Markovian interaction*

Considering an exactly solvable model of a system interacting with an environment, as in equation (1) where now $\hat{H}_0 = \hbar(\omega_a \hat{a}^\dagger \hat{a} + \sum_m \omega_{b_m} \hat{b}_m^\dagger \hat{b}_m)$ is the free-field Hamiltonian and $\hat{H}_\mathrm{I} = \hbar \sum_m G_m \hat{a}^\dagger \hat{b}_m + G_m^* \hat{a} \hat{b}_m^\dagger$ is the interaction Hamiltonian, where $\hat{b}_m$ ($\hat{a}$) is the bosonic annihilation operator for environmental mode b_m (system mode a) and G_m is the coupling constant for the interaction between them. It should be noted that G_m may contain an intrinsic time dependence depending on its form in which case G_m should be simply replaced with the form $G_m(t)$.

Beyond the Markovian approximation, the environment is able to preserve the coherent information of the system within its relaxation time, where $\tau_\mathrm{R} \gtrsim \tau_\mathrm{D}$. Hence, once the environment is perturbed by the system, it can memorize a part of the system information during τ_R which in turn is fed back to the system during another perturbation within τ_R. Given that the system and environment are in resonance, that is $\omega_a = \omega_{b_m}$, then it is possible to decompose the total evolution into interaction and free evolutions as $[\hat{H}_0, \hat{H}_\mathrm{I}] = 0$;

$$\exp\left(\frac{1}{\mathrm{i}\hbar}\hat{H}_\mathrm{T}\tau\right) = \exp\left(\frac{1}{\mathrm{i}\hbar}\hat{H}_0\tau\right)\exp\left(\frac{1}{\mathrm{i}\hbar}\hat{H}_\mathrm{I}\tau\right). \tag{19}$$

This just leaves the evolution due to the interaction Hamiltonian to be investigated, which on keeping N finite is

$$\hat{U}_\mathrm{I}(\tau) = \exp\left(-\mathrm{i}\sum_{m=1}^{N}\left[g_m^*(\tau)\hat{a}\hat{b}_m^\dagger + g_m(\tau)\hat{a}^\dagger\hat{b}_m\right]\right). \tag{20}$$

For brevity the notation $g_m(\tau) = G_m\tau$ has been used with a similar expression for $g_m^*(\tau)$. This evolution is such that the bosonic operators for the system mode a and environmental modes b_m are converted as follows:

$$\hat{U}_\mathrm{I}^\dagger(\tau)\hat{a}\hat{U}_\mathrm{I}(\tau) = \hat{a}\cos\theta - \frac{\mathrm{i}\sin\theta}{\theta}\left(\sum_l g_l(\tau)\hat{b}_l\right), \tag{21}$$

$$\hat{U}_\mathrm{I}^\dagger(\tau)\hat{b}_m\hat{U}_\mathrm{I}(\tau) = -\frac{\mathrm{i}\,\hat{a}g_m^*(\tau)\sin\theta}{\theta} + \hat{b}_m + \frac{g_m^*(\tau)(\cos\theta - 1)}{\theta^2}\left(\sum_l g_l(\tau)\hat{b}_l\right), \tag{22}$$

where $\theta^2 = \sum_l g_l(\tau)g_l^*(\tau)$, with similar results for the creation operators.

3.2 *Decomposition of the interaction Hamiltonian*

In order to try to understand the non-Markovian interaction, the interaction Hamiltonian was examined by investigating how $\hat{U}_{\mathrm{I}}(\tau)$ alters the modes a and b_m, as in equations (21) and (22). We find that $\hat{U}_{\mathrm{I}}(\tau)$ is decomposable as follows:

$$\hat{U}_{\mathrm{I}}(\tau) = \hat{U}_R \hat{B}_{b_N,a} \hat{U}'_R, \tag{23}$$

where $\hat{U}_R = \otimes_{j=1}^{N-1} \hat{R}_{b_j} \hat{B}_{b_{j+1},b_j}(g^*_{j+1}/\theta_{j+1} : \theta_j/\theta_{j+1})$, $\hat{B}_{b_N,a} = \exp[\mathrm{i}\,\theta_N(\hat{a}\hat{b}^\dagger_N + \hat{a}^\dagger \hat{b}_N)]$ and $\hat{U}'_R = \otimes_{j=1}^{N-1} \hat{R}_{b_{N-j}} \hat{B}_{b_{N-j+1},b_{N-j}}(g_{N-j+1}/\theta_{N-j+1} : \theta_{N-j}/\theta_{N-j+1})$, where $\hat{R}_{b_j} = e^{\mathrm{i}\pi \hat{b}^\dagger_j \hat{b}_j}$ is a rotator introducing a π phase shift on mode b_j, $\hat{B}_{b_{j+1},b_j}(t : r)$ is a beam splitter such that $\hat{B}^\dagger_{b_{j+1},b_j} \hat{b}_{j+1} \hat{B}_{b_{j+1},b_j} = t\hat{b}_{j+1} + r\hat{b}_j$ whilst $\hat{B}^\dagger_{b_{j+1},b_j} \hat{b}_l \hat{B}_{b_{j+1},b_j} = t^*\hat{b}_j - r^*\hat{b}_{j+1}$. Throughout, $\theta_j^2 = \sum_{l=1}^{j} g_l(\tau) g^*_l(\tau)$, $1 \leqslant j \leqslant N$. For the commutation relations from the output modes from a beam splitter to be satisfied, $|r|^2 + |t|^2 = 1$ and $r^*t + rt^* = 0$ must be true. For the above operators, these conditions will always be true provided that the coupling g_1 is real whilst all others are imaginary.

What this decomposition details is that all environmental modes b_j interact in such a way that they 'share' their information. The environmental mode that contains information from all others (b_N) then interacts with the system. Through this single system–environment interaction, the system gains information about all the environmental modes. In turn, the information about the system which the single mode b_N gains is then distributed back to all other environmental modes via the rotator–beam-splitter pairs of operators, only now working from the Nth to the first mode. This is a significant fact, as the system information is distributed by the environmental modes themselves, even though only one system–environment interaction takes place. It is known that any combination of rotators and beam splitters does not in itself bring about entanglement in the output fields, given classical input fields [13].

Let us assume that the environment is in thermal equilibrium as in section 2. It is obvious that a thermal field is rotation invariant and that beam splitting of two thermal fields at the same temperature does not change their states. In this case, $\hat{U}'_R$ does not alter the state of the environment. As $\hat{U}_R$ acts only on the environment, the system is affected only by the single beam-splitter operation $\hat{B}_{b_N,a} = \exp[\mathrm{i}\theta_N(\hat{a}\hat{b}^\dagger_N + \hat{a}^\dagger \hat{b}_N)]$. A Markovian environment could be modelled by an infinite array of beam splitters [5], which when transferred into the effective variable approach abbreviated down to a single beam-splitter interaction. The above decomposition, in the case of a thermal environment, details the exact same conclusion, namely that one beam-splitter interaction between the system and the environment brings about their joint correlations and consequently the decoherence of the system.

4. Remarks

In summary, it has been shown how through the use of the effective Hamiltonian approach, complete positivity of the reduced dynamics of a master equation can be guaranteed. This was achieved by deriving the master equation for a system using both the standard and the effective Hamiltonian approaches. The standard approach gave complete positivity on the short timescale (weak-coupling limit) and was then incorporated into the latter approach as boundary conditions upon the master equation, thus resulting in a superoperator solution valid for all time and completely positive.

Secondly, in the case of resonant interaction, the evolution due to the system–environment interaction was decomposed into an intricate and yet physically understandable combination of rotators and beam splitters. Within this non-Markovian interaction, $\hat{U}'_R$ and $\hat{U}_R$ described how the environment 'share' their information both before contact with the system and afterwards so that the systems information is distributed throughout the environment. However, in the case when the environment is in thermal equilibrium, then only the action of $\hat{B}_{b_N,a}$ affects the system. This single beam-splitter interaction is the same result as produced when considering a system–environment interaction under the Markovian approximation and effective variables are used.

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